OXFORD INTERNATIONAL AQA EXAMINATIONS

INTERNATIONAL A-LEVEL FURTHER MATHEMATICS

(9665)

Further Pure Mathematics Unit 2 Mark Scheme

Specimen 2018

Principal Examiners have prepared these mark schemes for specimen papers. These mark schemes have not, therefore, been through the normal process of standardising that would take place for live papers.

Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct and answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
OE	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
$-x$ **EE**	Deduct x marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
c	Candidate
sf	Significant figure(s)
dp	Decimal place(s)

No method shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be ontained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Q	Answer	Marks	Comments
1	Area = $\frac{1}{2}\int \left(2\sqrt{1+\tan\theta}\right)^2 \,(\mathrm{d}\theta)$	M1	Use of $\frac{1}{2}\int r^2 \,(\,\mathrm{d}\theta)$
	$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{0} 4(1+\tan\theta)\ \mathrm{d}\theta$	B1	Correct limits. If any contradiction use the limits at the substitution stage
	$= 2\left[\theta+\ln\sec\theta\right]_{-\frac{\pi}{4}}^{0}$	B1	$\int k(1+\tan\theta)(\mathrm{d}\theta) = k(\theta+\ln\sec\theta)$ ACF ft on their k
	$= 2\left\{0-\left[-\frac{\pi}{4}+\ln\sec\left(-\frac{\pi}{4}\right)\right]\right\}$ $= 2\left(\frac{\pi}{4}-\ln\sqrt{2}\right) = \frac{\pi}{2}-2\ln\sqrt{2} = \frac{\pi}{2}-\ln 2$	A1	CSO AG
	Total	**4**	

Q	Answer	Marks	Comments
2(a)	$\frac{(e^x+e^{-x})}{2}\frac{(e^y+e^{-y})}{2}-\frac{(e^x-e^{-x})}{2}\frac{(e^y-e^{-y})}{2}$	M1A1	M0 if sinh and cosh confused M1 for formula quoted correctly
	Correct expansions	A1	Use of e^{xy} A0
	$=\frac{1}{2}\left(e^{x-y}+e^{-(x-y)}\right)=\cosh(x-y)$	A1	AG
2(b)(i)	$\cosh(x-\ln 2)=\cosh x\cosh(\ln 2)$ $-\sinh x\sinh(\ln 2)$	M1	
	$\left.\begin{array}{l}\cosh(\ln 2)=\frac{5}{4}\\ \sinh(\ln 2)=\frac{3}{4}\end{array}\right\}$ any method	B1	Both correct Alternative $\frac{e^{x-\ln 2}+e^{-x+\ln 2}}{2}=\frac{e^x-e^{-x}}{2}$ M1 $e^{x-\ln 2}=\frac{e^x}{2}$ or $e^{-x+\ln 2}=2e^{-x}$ used B1 $e^x=\sqrt{6}$ A1 $\tanh x=\frac{5}{7}$ A1
	$\frac{5}{4}\cosh x=\frac{7}{4}\sinh x$	A1ft	
	$\tanh x=\frac{5}{7}$	A1	AG
2(b)(ii)	$x=\frac{1}{2}\ln\left(\frac{1+\frac{5}{7}}{1-\frac{5}{7}}\right)$ or $\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{5}{7}$	M1	
	$=\frac{1}{2}\ln 6$	A1	
	Total	10	

Q	Answer	Marks	Comments
3(a)	$\alpha+\beta+\gamma=2$	B1	
3(b)(i)	α is a root and so satisfies the equation	E1	
3(b)(ii)	$\sum\alpha^3-2\sum\alpha^2+p\sum\alpha+30=0$	M1A1	
	Substitution for $\sum\alpha^3$ and $\sum\alpha$	m1	
	$\sum\alpha^2=p+13$	A1	AG
3(b)(iii)	$\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$ used	M1	do not allow this M mark if used in (b)(ii)
	$p=-3$	A1	AG
3(c)(i)	$\mathrm{f}\left(-2\right)=0$	M1	
	$\alpha=-2$	A1	
3(c)(ii)	$\left(z+2\right)\left(z^2-4z+5\right)=0$	M1	For attempting to find quadratic factor
	$z=\dfrac{4\pm\sqrt{-4}}{2}$	m1	Use of formula or completing the square m0 if roots are not complex
	$=2\pm\mathrm{i}$	A1	CAO
	Total	**13**	

Q	Answer	Marks	Comments
4(a)	det M = $\begin{vmatrix} k & 3 \\ k & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ k & 1 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ k & 3 \end{vmatrix}$ $= (k - 3k) - 3(4 - 2k) + 2(12 - 2k)$	M1	Correct expansion by row or column
	= 12	A1	CAO
	(Constant/Independent of k and) therefore can never equal zero – hence non singular	E1	Explanation – must refer to non-zero answer and M1A1 must have been scored.
4(b)	$\begin{bmatrix} -2k & 3 & k \\ 2k-4 & -3 & 8-k \\ 12-2k & 3 & k-12 \end{bmatrix}$	M1 A(2, 1)	M1 Cofactor matrix – one full row or column correct A2 all entries correct A1 at least six entries correct
	$\mathbf{M}^{-1} = \frac{1}{12}\begin{bmatrix} -2k & 2k-4 & 12-2k \\ 3 & -3 & 3 \\ k & 8-k & k-12 \end{bmatrix}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 25 \\ 3 \\ 2 \end{pmatrix}$	m1 A1ft	m1 Divide by determinant and transpose their matrix Follow through their determinant answer in part (a) – must be non-zero
4(b)(ii)	$= \frac{1}{12}\begin{pmatrix} -50k + 6k - 12 + 24 - 4k \\ 75 - 9 + 6 \\ 25k + 24 - 3k + 2k - 24 \end{pmatrix}$	M1A1ft	M1 Attempt at $\mathbf{M}^{-1}\mathbf{v}$ one of their components correct - can be unsimplified A1 Two of their components correct - can be unsimplified. Follow through their $\mathbf{M}^{-1}$
	$= \frac{1}{12}\begin{pmatrix} 12 - 48k \\ 72 \\ 24k \end{pmatrix}$ $= \begin{pmatrix} 1 - 4k \\ 6 \\ 2k \end{pmatrix}$	A1	Fully correct and simplified – CSO
	Hence $x = 1 - 4k$ $y = 6$ $z = 2k$	A1	Any method which does not use $\mathbf{M}^{-1}\mathbf{v}$ scores zero marks
	Total	12	

Q	Answer	Marks	Comments
5(a)	$\frac{dy}{dx}+\frac{\sec^2 x}{\tan x}y=\tan x$ IF is exp ($\int \frac{\sec^2 x}{\tan x}\,dx$)	M1	and with integration attempted
	$= e^{\ln(\tan x)} = \tan x$	A1	AG Be convinced
5(b)	$\tan x\frac{dy}{dx}+(\sec^2 x)y=\tan^2 x$ $\frac{d}{dx}[y\tan x]= \tan^2 x$	M1	LHS as differential of $y \times$ IF PI
	$y \tan x = \int \tan^2 x\, dx$	A1	
	$y \tan x = \int (\sec^2 x - 1)\, dx$	m1	Using $\tan^2 x = \pm \sec^2 x \pm 1$ PI or other valid methods to integrate $\tan^2 x$
	$y \tan x = \tan x - x$ $(+c)$	A1	Correct integration of $\tan^2 x$; condone absence of $+c$.
	$3\tan\frac{\pi}{4}=\tan\frac{\pi}{4}-\frac{\pi}{4}+c$	m1	Boundary condition used in attempt to find value of c
	$c = 2+\frac{\pi}{4}$ so $y \tan x = \tan x - x + 2 + \frac{\pi}{4}$ $y = 1 + (2 - x + \frac{\pi}{4}) \cot x$	A1	ACF
	Total	13	

Q	Answer	Marks	Comments
6	Char. Equ is $\lambda^2 - 8\lambda - 9 = 0$	M1	Attempted
	Quadratic solved to get two roots	m1	
	$\Rightarrow\ \lambda = 9, -1$	A1	
	Subst^g, λ back (at least once)	M1	
	$\Rightarrow\ \lambda = 9$ has evecs $\alpha\begin{bmatrix}1\\1\end{bmatrix}$ $\lambda = -1 \Rightarrow x + y = 0$	A1	any $\alpha \neq 0$
	$\Rightarrow\ \lambda = -1$ has evecs $\beta\begin{bmatrix}1\\-1\end{bmatrix}$	A1	any $\beta \neq 0$
	Total	6	

Q	Answer	Marks	Comments
7(a)	$\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}-3y=3x-8e^{-3x}$ P. Integral $y_{PI}=a+bx+cxe^{-3x}$ $y'_{PI}=b+ce^{-3x}-3cxe^{-3x}$	M1	± pe−3x ± qxe−3x
	$y''_{PI}=-6ce^{-3x}+9cxe^{-3x}$ $-6ce^{-3x}+9cxe^{-3x}+2b+2ce^{-3x}-6cxe^{-3x}$ $-3a-3bx-3cxe^{-3x}=3x-8e^{-3x}$	M1	Substitution into LHS of DE
	$-3b=3$; $2b-3a=0$; $-4c=-8$	m1	Dep on 2nd M only Equating coeffs to obtain at least two of these correct eqns; PI by correct values for at least two constants
	$b=-1$; $c=2$; $a=-\dfrac{2}{3}$	A2, 1, 0	Dep on M1M1m1 all awarded A1 if any two correct; A2 if all three correct but do not award the 2nd A mark if terms in xe^{-3x} were incorrect in the M1 line
7(b)	Aux. eqn. $m^2+2m-3=0$ $(m+3)(m-1)=0$	M1	Factorising or using quadratic formula oe PI by correct two values of 'm' seen/used
	$(y_{CF}=)Ae^{-3x}+Be^{x}$	A1	their *CF* + their PI with 2 arbitrary constants, non-zero values for *a*, *b* and *c* and no trig or ln terms in their *CF*
	$(y_{GS}=)Ae^{-3x}+Be^{x}-\dfrac{2}{3}-x+2xe^{-3x}$	B1ft	
7(c)	$x=0,\ y=1\ \Rightarrow 1=A+B-\dfrac{2}{3}$ $\dfrac{dy}{dx}=-3Ae^{-3x}+Be^{x}-1+2e^{-3x}-6xe^{-3x}$	B1ft	Only ft if previous B1ft has been awarded
	As $x\to\infty$, ($e^{-3x}\to 0$ and) $xe^{-3x}\to 0$	E1	Must treat xe^{-3x} separately
	(As $x\to\infty$, $\dfrac{dy}{dx}\to -1$ so) $B=0$ When $B=0$, $1=A-\dfrac{2}{3}\ \Rightarrow A=\dfrac{5}{3}$	B1	B= 0, where B is the coefficient of e^x.
	$y=\dfrac{5}{3}e^{-3x}-\dfrac{2}{3}-x+2xe^{-3x}$	A1	
	Total	**12**	

Q	Answer	Marks	Comments
8(a)	$y=(4+\sin x)^{1/2}$ so $y^2 = 4+\sin x$ $2y\dfrac{dy}{dx} = \cos x$	M1	$\dfrac{d}{dx}(y^2)=2y\dfrac{dy}{dx}$
	$y\dfrac{dy}{dx}=\dfrac{1}{2}\cos x$	A1	Chain rule
	Alternative $\dfrac{dy}{dx}=\dfrac{1}{2}(4+\sin x)^{-1/2}(\cos x)$	(M1)	
	$y\dfrac{dy}{dx}=\dfrac{1}{2}\cos x$	(A1)	
8(b)	$y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=-\dfrac{1}{2}\sin x$	M1	Correct differentiation of $y\dfrac{dy}{dx}$
	When $x = 0$, $y = 2$, $\dfrac{dy}{dx}=\dfrac{1}{4}$, $2\dfrac{d^2y}{dx^2}+\left(\dfrac{1}{4}\right)^2=0$	A1ft	Ft on RHS of M1 line as $k\sin x$
	$y\dfrac{d^3y}{dx^3}+\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}=-\dfrac{1}{2}\cos x$	m1 A1	Correct LHS
	When $x = 0$, $2\dfrac{d^3y}{dx^3}+3\left(\dfrac{1}{4}\right)\left(-\dfrac{1}{32}\right)=-\dfrac{1}{2}\Rightarrow\dfrac{d^3y}{dx^3}=-\dfrac{61}{256}$	A1	
	Alternative $\dfrac{d^2y}{dx^2}=-\dfrac{1}{4}(4+\sin x)^{-3/2}(\cos^2 x)+\dfrac{1}{2}(4+\sin x)^{-1/2}(-\sin x)$	(M1)	Sign and numerical coeffs errors only. ACF
	$\dfrac{d^3y}{dx^3}=\dfrac{3}{8}(4+\sin x)^{-2.5}(\cos^3 x)-\dfrac{1}{4}(4+\sin x)^{-1.5}(-2\cos x\sin x)$ $-\dfrac{1}{4}(4+\sin x)^{-1.5}(\cos x)(-\sin x)-\dfrac{1}{2}(4+\sin x)^{-0.5}\cos x$	(A1)	Sign and numerical coeffs errors only. ACF
	When $x = 0$, $\dfrac{d^3y}{dx^3}=\dfrac{3}{8}\times\dfrac{1}{32}-\dfrac{1}{2}\times\left(\dfrac{1}{2}\right)=-\dfrac{61}{256}$	(A1)	CSO

Q	Answer	Marks	Comments
9(a)	$7+4x-2x^2 = 9-2(x-1)^2$	M1A1	
9(b)	Put $u=\sqrt{2}(x-1)$	M1	Allow $u=k(x-1)$ any k
	$\mathrm{d}u=\sqrt{2}\,\mathrm{d}x$	A1ft	
	$\mathrm{I}=\frac{1}{\sqrt{2}}\int\frac{\mathrm{d}u}{\sqrt{9-u^2}}$	A1ft	ft on (a) ie $\frac{1}{\sqrt{b}}\int\frac{du}{\sqrt{a-u^2}}$
	$=\frac{1}{\sqrt{2}}\sin^{-1}\frac{u}{3}$	A1	for $\sin^{-1}\frac{u}{p}$
	Change limits or replace u	m1	provided $\sin^{-1}$
	$=\frac{\pi}{4\sqrt{2}}$ or $\frac{\pi\sqrt{2}}{8}$	A1	CAO
	Alternative If integration is attempted without substitution: $\sin^{-1}$	(M1)	
	$\frac{1}{\sqrt{2}}$	(A1ft)	
	$(x-1)$	(A1)	
	$\frac{\sqrt{2}}{3}$	(A1ft)	
	Substitution of limits	(m1)	
	$\frac{\pi}{4\sqrt{2}}$	(A1)	CAO
	Total	**8**	

Q	Answer	Marks	Comments
10	Assume result true for $n = k$ Then $u_{k+1} = \dfrac{3}{4 - \left(\dfrac{3^{k+1} - 3}{3^{k+1} - 1}\right)}$	M1	
	$= \dfrac{3(3^{k+1} - 1)}{4(3^{k+1} - 1) - (3^{k+1} - 3)}$	A1	
	$4 \times 3^{k+1} - 3^{k+1} = 3^{k+2}$	A1	Clearly shown
	$u_{k+1} = \dfrac{3^{k+2} - 3}{3^{k+2} - 1}$	A1	
	$n = 1 \quad \dfrac{3^2 - 3}{3^2 - 1} = \dfrac{3}{4} = u_1$	B1	
	Induction proof set out properly	E1	Must have earned previous 5 marks
	Total	**6**	

Q	Answer	Marks	Comments
11(a)	$\dfrac{\mathrm{d}x}{\mathrm{d}t} = \sec t - \cos t$	B1, B1	use of FB to obtain $\sec t$; if done from first principles, allow B1 when $\sec t$ is arrived at□
	Use of $1 - \cos^2 t = \sin^2 t$	M1	
	$\dfrac{\mathrm{d}x}{\mathrm{d}t} = \sin t \tan t$	A1	AG
11(b)	$\dot{x}^2 + \dot{y}^2 = \sin^2 t \tan^2 t + \sin^2 t$ Use of $1 + \tan^2 t = \sec^2 t$	M1 A1 m1	sign error in $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ A0□
	$\sqrt{\dot{x}^2 + \dot{y}^2} = \tan t$	A1ft	Ft sign error in $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ □
	$\int_0^{\frac{\pi}{3}} \tan t \ \mathrm{d}t = \left[\ln \sec t\right]_0^{\frac{\pi}{3}}$	A1ft	Ft sign error in $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ □
	$= \ln 2$	A1	CAO
	Total	**8**	

Q	Answer	Marks	Comments
12(a)	$\left.\begin{array}{l}\text{direction ratios of line} = p:\ 3:-1\\ \text{normal to plane} \quad = 1:\ \ 1:\ \ 2\end{array}\right\}$ not equal	B1	Accept not parallel or showing vector product is non zero
12(b)	$x = 3 + pt$ $y = q + 3t$ $z = 1 - t$	M1	Parametric form seen
	Meets plane $\Rightarrow (5 + q) + t(p + 1) = 10$ $\Rightarrow 3 + pt + q + 3t + 2(1 - t) = 10$	A1	Correct substitution in plane
	Within plane $\Rightarrow q = 5,\ p = -1$	M1A1	M1 - Finding one correct value A1 - Both values correct
	Alternative Point (3, q, 1) is common to line and plane Hence $\begin{pmatrix}3\\q\\1\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix} = 10$ which gives $q = 5$	(M1A1)	Uses common point to find q
	Another point common to both is $(3 + p, 8, 0)$ Hence $\begin{pmatrix}3+p\\8\\0\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix} = 10$ which gives $p = -1$	(M1A1)	Use of second point and value of q to find p or consideration of scalar product $\begin{pmatrix}p\\3\\-1\end{pmatrix}.\begin{pmatrix}1\\1\\2\end{pmatrix} = 0$

Q	Answer	Marks	Comments
12(c)(i)	$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} p \\ 3 \\ -1 \end{pmatrix}$ Let α be angle between normal and direction ratios (plane) (line) $\mathbf{n}.\mathbf{d} = p+1$	M1	n . d correct
	$\sin\theta = \dfrac{1}{\sqrt{6}} \Rightarrow \cos\alpha = \dfrac{\pm 1}{\sqrt{6}}$	B1	Correct cos α stated or implied
	$\Rightarrow \dfrac{p+1}{\sqrt{6}\sqrt{p^2+10}} = \dfrac{\pm 1}{\sqrt{6}}$ $\Rightarrow (p+1)^2 = p^2+10$ $\Rightarrow p^2+2p+1 = p^2+10$	m1A1	Forming equation connecting all relevant parts and attempting to solve for p (condone missing ±) Dependent on first M1 - fully correct for A1
	$\Rightarrow 2p = 9$ giving $p = 4.5$	A1	CAO
	Alternative $\lvert\mathbf{n}\times\mathbf{d}\rvert = \sqrt{49+(1+2p)^2+(3-p)^2}$	(M1)	n . d correct
	$\sin\theta = \dfrac{1}{\sqrt{6}} \Rightarrow \sin\alpha = \dfrac{\sqrt{5}}{\sqrt{6}}$	(B1)	Correct cos α stated or implied
	$\dfrac{\sqrt{49+(1+2p)^2+(3-p)^2}}{\sqrt{6}\sqrt{p^2+10}} = \dfrac{\sqrt{5}}{\sqrt{6}}$	(m1A1)	Forming equation connecting all relevant parts and attempting to solve for p (condone missing ±) Dependent on first M1 – fully correct for A1
	Leading to $p = 4.5$	(A1)	CAO
12(c)(ii)	$z = 2 => t = -1 => x = -1.5$ $p = 4.5$ $y = q-3$ $\Rightarrow -1.5 + q - 3 + 4 = 10$	M1	Attempt to form an equation for q using t = –1
	$q = 10.5$	A1	CAO
	Total	**12**	

Q	Answer	Marks	Comments
13(a)(i)	$1+\sqrt{3}\,i = 2e^{\frac{\pi i}{3}}$ oe	B1	
	$1 - \mathrm{I} = \sqrt{2}\,e^{\frac{\pi i}{4}}$ oe	B1	
		B1	If both in the correct form
13(a)(ii)	$2^{\frac{21}{2}}$ or equivalent single expression □	B1ft	No decimals; must include fractional powers
	Raising and adding powers of e	M1	
	$\frac{17\pi}{12}$ or equivalent angle	A1ft	Denominators of angles must be different
13(b)	$z = \sqrt[3]{2^{10}\sqrt{2}}\,e^{\frac{17\pi i}{36}+\frac{2k\pi i}{3}}$	M1	
	$\sqrt[3]{2^{10}\sqrt{2}} = 8\sqrt{2}$	B1	CAO
	$\theta = \frac{17\pi}{36}, -\frac{7\pi}{36}, -\frac{31\pi}{36}$	A2 1F	Correct answers outside range: deduct 1 mark only
		Total	**10**