OXFORD
INTERNATIONAL
AQA EXAMINATIONS

INTERNATIONAL AS FURTHER MATHS

(FM01)

Unit 1: Further Pure Mathematics

Example responses with commentary

For teaching from September 2016 onwards
For AS exams in May/June 2018 onwards

This guide includes some examples of student responses to a selection of questions from the summer 2018 FM01 unit.

The question parts are reproduced, along with the final mark scheme, student responses and a commentary from the Lead Examiner on each of the students' answers.

QUESTION

02

2 It is given that $z = x + \mathrm{i}y$, where x and y are real.

Also, $\dfrac{3-2\mathrm{i}}{z} = a + \mathrm{i}b$, where a and b are real.

Find a and b in terms of x and y.

[3 marks]

Answer

MARK SCHEME

Q	Answer	Mark	Total	Comments
2	$\frac{3-2i}{x+iy}\times\frac{x-iy}{x-iy}$	M1	3	or $ax+by=3$ and $ay+bx=-2$
	$a=\frac{3x-2y}{x^2+y^2}$	A1		
	$b=\frac{-2x-3y}{x^2+y^2}$	A1		
Total	3			

STUDENT A

2 It is given that $z = x + iy$, where x and y are real.

Also, $\frac{3-2i}{z} = a + ib$, where a and b are real.

Find a and b in terms of x and y.

[3 marks]

$\frac{3-2i}{x+iy} = a + ib$

$\frac{3}{x+iy} - \frac{2i}{x+iy}$

$\frac{(3-2i)(x-iy)}{(x+iy)(x-iy)} = \frac{3x - 3iy - 2ix + 2i^2y}{x^2 - iyx + iyx - i^2y}$

$= \frac{3x - 2y - (3y + 2x)i}{x^2 + y^2}$

$a = \frac{3x - 2y}{x^2 + y^2}$ $\quad b = \frac{-(3y + 2x)}{x^2 + y^2}$

Answer

EXAMINER COMMENTARY

This student uses a direct method – multiplying the numerator and denominator by the complex conjugate of the denominator.
This quickly leads to the correct solution.
Mark 3/3

STUDENT B

2 It is given that $z = x + iy$, where x and y are real.

Also, $\frac{3-2i}{z} = a + ib$, where a and b are real.

Find a and b in terms of x and y.

[3 marks]

$$\frac{3-2i}{x+iy} = a+ib$$

$$\Rightarrow 3-2i = (a+ib)(x+iy)$$

$$\Rightarrow 3-2i = ax + iay + ibx - by$$

$$\therefore ax - by = 3 \quad (1)$$

$$ay + bx = -2 \quad (2)$$

$$(1)\quad ax - 3 = by$$

$$\therefore b = \frac{ax-3}{y}$$

$$(2)\quad \cancel{\frac{a^2x - 3a}{y} + bx = -2}$$

$$(2)\quad ay + \frac{(ax-3)x}{y} = -2$$

$$\therefore ay^2 + ax^2 - 3x = -2y$$

$$\Rightarrow a(y^2+x^2) = 3x - 2y$$

$$a = \frac{3x-2y}{y^2+x^2}$$

$$\therefore b = \frac{\frac{3x-2y}{y^2+x^2} - 3}{y}$$

$$= \frac{3x - 2y - 3y^2 - 3x^2}{y^3 + x^2y}$$

$$= \frac{3x(1-x) - y(2+3y)}{y(y^2+x^2)}$$

Answer $a = \frac{3x-2y}{x^2+y^2}$, $\cancel{b = 3x(1-x) - y(2+3y)}$

$$b = \frac{3x(1-x) - y(2+3y)}{y(y^2+x^2)}$$

EXAMINER COMMENTARY

This student performs a multiplication and then equates real and imaginary parts.
This is a correct method, and the student successfully finds a in terms of x and y.
However, when attempting to find b, they slip up and use $b = \frac{a-3}{y}$ instead of $b = \frac{ax-3}{y}$ as intended.
Mark 2/3

COMPARISON

It is recommended to use the method employed in script 1, because this involves less working and there is therefore less opportunity to make a mistake.

QUESTION

03

3 (a) Given that $f(r) = \dfrac{1}{r+2}$, show that

$$f(r) - f(r+1) = \frac{1}{(r+2)(r+3)}$$

[1 mark]

3 (b) Use the method of differences to:

3 (b) (i) find the exact value of

$$\sum_{r=11}^{30} \frac{1}{(r+2)(r+3)}$$

[4 marks]

MARK SCHEME

Q	Answer	Mark	Total	Comments
3(a)	$\dfrac{(r+3)-(r+2)}{(r+2)(r+3)} = \dfrac{1}{(r+2)(r+3)}$	B1	1	
3(b)(i)	$\sum_{r=11}^{30} \dfrac{1}{(r+2)(r+3)} = f(11) - f(12)$ $+f(12) - f(13)$ $+\ldots\ldots\ldots\ldots$ $+f(29) - f(30)$ $+f(30) - f(31)$	M1 A1	4	
	$= f(11) - f(31)$ or $= \dfrac{1}{13} - \dfrac{1}{33}$	A1		
	$= \dfrac{20}{429}$	A1		
3(b)(ii)	$\sum_{r=18}^{n} \dfrac{1}{(r+2)(r+3)} = f(18) - f(n+1)$	M1	4	PI
	$= \dfrac{1}{20} - \dfrac{1}{n+3}$	A1		
	$\lim_{n\to\infty}\left(\dfrac{1}{20} - \dfrac{1}{n+3}\right)$	E1		For correct explanation
	$= \dfrac{1}{20}$	B1		
Total	9			

STUDENT A

3 (a) Given that $f(r) = \frac{1}{r+2}$, show that

$$f(r) - f(r+1) = \frac{1}{(r+2)(r+3)}$$

[1 mark]

$$\frac{1}{r+2} - \frac{1}{r+1+2} = \frac{1}{r+2} - \frac{1}{r+3}$$

$$= \frac{\cancel{r}+3-\cancel{r}-2}{(r+2)(r+3)} = \frac{1}{(r+2)(r+3)}$$

3 (b) Use the method of differences to:

3 (b) (i) find the exact value of

$$\sum_{r=11}^{30} \frac{1}{(r+2)(r+3)}$$

[4 marks]

$\frac{1}{13}$ +

$\frac{1}{13} + \frac{1}{14}$

$\frac{1}{14}$ −

$\cancel{\frac{1}{30+2}}$ −

$\left(\frac{1}{13} - \frac{1}{14}\right) + \left(\frac{1}{14} - \frac{1}{15}\right) + \left(\frac{1}{15} - \frac{1}{16}\right) \ldots \left(\frac{1}{31} - \frac{1}{32}\right) + \left(\frac{1}{32} - \frac{1}{33}\right)$

$\frac{1}{13} - \frac{1}{33} = \frac{20}{429}$

Answer $\frac{20}{429}$

3 (b) (ii) show that

$$\sum_{r=18}^{\infty} \frac{1}{(r+2)(r+3)} = \frac{1}{m}$$

where m is an integer.

[4 marks]

$\lim_{\infty \to n} \quad \frac{1}{n+2} - \frac{1}{20}$

$\sum_{r=19}^{n} \quad \frac{1}{n+2} - \frac{1}{n+3} \qquad \left(\frac{1}{20} - \cancel{\frac{1}{21}}\right) + \left(\cancel{\frac{1}{21}} - \frac{1}{22}\right) + \dots$

$\dots + \left(\frac{1}{(n-1)+2} - \cancel{\frac{1}{(n-1)+3}}\right) + \left(\cancel{\frac{1}{n+2}} - \frac{1}{(n+3)}\right)$

$\frac{1}{20} - \frac{1}{n+3} \qquad \frac{1}{n+3} \to 0$ as $n \to \infty$

$\therefore \lim_{n \to \infty} \sum_{r=19}^{n} \frac{1}{(r+2)(r+3)} = \frac{1}{20}$

$m = 20$

EXAMINER COMMENTARY

Parts (a) and (b)(i) are done correctly.
In part (b)(ii) the limiting process as n tends to infinity is shown clearly.
Marks 1/1, 4/4, 4/4

STUDENT B

3 (a) Given that $f(r) = \dfrac{1}{r+2}$, show that

$$f(r) - f(r+1) = \frac{1}{(r+2)(r+3)}$$

[1 mark]

$$\frac{1}{r+2} - \frac{1}{r+3} = \frac{(r+3)}{(r+2)(r+3)} - \frac{(r+2)}{(r+2)(r+3)}$$

$$= \frac{(r+3)-(r+2)}{(r+2)(r+3)} = \frac{1}{(r+2)(r+3)}$$

3 (b) Use the method of differences to:

3 (b) (i) find the exact value of

$$\sum_{r=11}^{30} \frac{1}{(r+2)(r+3)}$$

[4 marks]

$r = 11 \rightarrow 30$

$$\cancel{= \overset{11}{\frac{1}{(13)(14)}} + \overset{12}{\frac{1}{(14)(15)}} + \overset{13}{\frac{1}{(15)(16)}} \ldots \frac{1}{(31)(32)} + \frac{1}{(32)(33)}}$$

$f(r) - f(r+1)$

$$= \frac{1}{13} - \cancel{\frac{1}{14}} + \cancel{\frac{1}{14}} - \cancel{\frac{1}{15}} + \cancel{\frac{1}{15}} \ldots \cancel{\frac{1}{32}} - \frac{1}{33}$$

$$= \frac{1}{13} - \frac{1}{33}$$

$$= \frac{20}{429}$$

Answer $\dfrac{20}{429}$

3 (b) (ii) show that

$$\sum_{r=18}^{\infty} \frac{1}{(r+2)(r+3)} = \frac{1}{m}$$

where m is an integer.

[4 marks]

$f(r) = \frac{1}{r+2}$

$f(r) - f(r+1)$

$= \frac{1}{20} - \cancel{\frac{1}{21}} + \cancel{\frac{1}{21}} - \cancel{\frac{1}{22}} \cdots \frac{1}{n+2} - \frac{1}{n+3}$

$= \cancel{\frac{1}{20}}$ as $n \rightarrow \infty$ $\frac{1}{n+2}$ and $\frac{1}{n+3} \rightarrow 0$

$\therefore = \frac{1}{20}$

EXAMINER COMMENATRY

Parts (a) and (b)(i) are done correctly.
In part (b)(ii) the value of ∞ is substituted into the relevant expression. The limiting process is not referred to.
Marks 1/1, 4/4, 3/4

COMPARISON

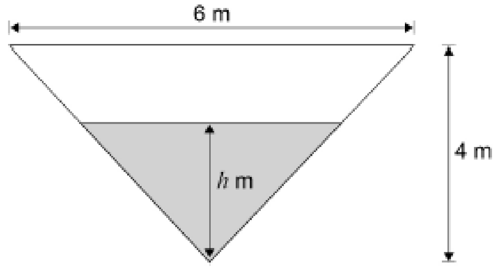
In order to gain full marks, students need to show the limiting process.

QUESTION

05

5 [The volume V of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the circular base of the cone and h is the height of the cone.]

A water tank in the shape of an inverted cone has height 4 metres and maximum diameter 6 metres. The cross-section of the tank is shown in the diagram.



The tank fills with water at a rate of 0.06 m^3 per minute.

At time t minutes after the tank starts to fill, the depth of water in the tank is h metres.

Find the rate at which h is increasing when $h = 2.5$

Give your answer in terms of π.

[8 marks]

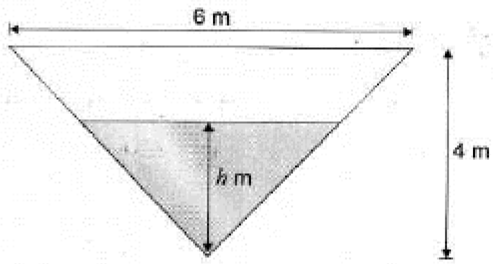
MARK SCHEME

Q	Answer	Mark	Total	Comments
5	$r = \frac{3}{4}h$	B1	8	
	$V = \frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h$ Or $\frac{3}{16}\pi h^3$	M1		
	$\frac{dV}{dh} = \frac{9}{16}\pi h^2$	M1 A1		M1 for differentiating "their" V A1 for correct derivative
	$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$	B1		
	$\frac{dV}{dt} = 0.06$ seen	B1		Seen anywhere
	$0.06 = \frac{9}{16}\pi(2.5)^2 \times \frac{dh}{dt}$	M1		Correct use of 0.06 and 2.5
	$\frac{dh}{dt} = \frac{32}{1875\pi}$ oe	A1		Allow $\frac{dh}{dt} = \frac{0.0170\dot{6}}{\pi}$
Total	8			

STUDENT A

5 [The volume V of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the circular base of the cone and h is the height of the cone.]

A water tank in the shape of an inverted cone has height 4 metres and maximum diameter 6 metres. The cross-section of the tank is shown in the diagram.



The tank fills with water at a rate of 0.06 m^3 per minute.

At time t minutes after the tank starts to fill, the depth of water in the tank is h metres.

Find the rate at which h is increasing when $h = 2.5$

Give your answer in terms of π.

[8 marks]

$\frac{r}{h} = \frac{4}{3}$ $\therefore r = \frac{4}{3}h$ $V = \frac{1}{3}\pi\left(\frac{4}{3}h\right)^2 h = \frac{16}{27}\pi h^3$

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$ $\frac{dV}{dh} = \frac{16}{27}\pi \cdot 3h^2 = \frac{16}{9}\pi h^2$

$\therefore \frac{dh}{dt} = \frac{9}{16\pi h^2} \cdot 0.06 = \frac{27}{800\pi h^2}$

When $h = 2.5$

$\frac{dh}{dt} = \frac{27}{800(\pi)(2.5)^2} = \frac{27}{5000\pi}$ m per minute.

EXAMINER COMMENTARY

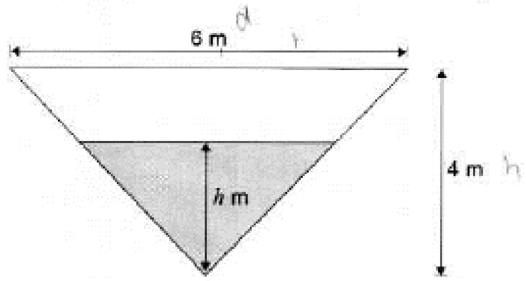
The student starts off by incorrectly stating that $\frac{r}{h} = \frac{4}{3}$ when it should be $\frac{r}{h} = \frac{3}{4}$ After that, everything is done correctly; V is found in terms of h only, the chain rule is used correctly and 0.06 is identified as being equal to $\frac{\mathrm{d}V}{\mathrm{d}t}$.
Finally the required value of h is substituted to give an answer that would be correct if their earlier equation [$\frac{r}{h} = \frac{4}{3}$] was correct.

Mark 6/8

STUDENT B

5 [The volume V of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the circular base of the cone and h is the height of the cone.]

A water tank in the shape of an inverted cone has height 4 metres and maximum diameter 6 metres. The cross-section of the tank is shown in the diagram.



The tank fills with water at a rate of 0.06 m^3 per minute.

At time t minutes after the tank starts to fill, the depth of water in the tank is h metres.

Find the rate at which h is increasing when $h - 2.5$

Give your answer in terms of π.

[8 marks]

the rate at which h $\Rightarrow \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\qquad \frac{dV}{dh} = \frac{1}{3} \times \pi \times r^2$

$0.06 = \cancel{\frac{1}{3}}\pi \cancel{36}^{12} \times \frac{dh}{dt}$ $\qquad$ (when $h = 2.5$ $V = 30\pi$)

$\frac{dh}{dt} = \frac{0.06}{12\pi} = \cancel{5\times10^{-3}}$ $\qquad \frac{30\pi}{0.06} = 500\pi$ min

$= 5\times10^{-3}\,\frac{1}{\pi}$ m/s $\qquad \frac{h}{t} = \frac{2.5}{500\pi} = \frac{1}{200\pi} = \frac{5\times10^{-3}}{\pi}$ m/s

$= \frac{5\times10^{-3}}{\pi}$ m/s

EXAMINER COMMENTARY

This students attempts to find $\frac{dV}{dh}$ by treating r as a constant, rather than by expressing r in terms of h and then expressing V in terms of h only.
Later on they substitute $r = 3$ into their expression for $\frac{dV}{dh}$
They could have gained a further mark if they had used $\frac{r}{h} = \frac{3}{4}$ at some stage.
As it is, they only gain the marks for using the chain rule correctly and equating $\frac{dV}{dt}$ to 0.06.

Mark 2/8

COMPARISON

This question was found difficult by students, probably because many of them did not see that as h increases, r is proportional to h, and r can be eliminated to give V in terms of h only.

The solution in script 7 follows this approach and is able to gain most of the marks despite an early error.

The solution in script 6 does not recognise that r is proportional to h, and therefore does not gain many marks.

QUESTION

06

6 (a) Find the value of

$$\sum_{r=1}^{45} (2r)^2$$

[2 marks]

Answer

6 (b) **Hence** find the sum of the squares of all the odd numbers from 1 to 89

[3 marks]

Answer

MARK SCHEME

Q	Answer	Mark	Total	Comments
6(a)	$4\sum_{r=1}^{45} r^2$	M1	2	PI
	125580	A1		
6(b)	$\sum_{r=1}^{90} r^2$ or 247065	B1	3	Seen anywhere
	247065 – 125580	M1		
	121485	A1		NMS 1/3
Total	5			

STUDENT A

6 (a) Find the value of

$$\sum_{r=1}^{45} (2r)^2$$

[2 marks]

$= \sum_{r=1}^{45} 4r^2 = 4\left(\sum_{r=1}^{45} r^2\right) = 4 \times \frac{45(46)(91)}{6}$

$= 125580$

Answer 125580

6 (b) Hence find the sum of the squares of all the odd numbers from 1 to 89

[3 marks]

$\sum_{r=1}^{\cancel{45}\,45} (2r-1)^2 = \sum_{r=1}^{\cancel{89}\,45} 4r^2 - 4r + 1$ $\left[\begin{array}{l} 2r-1=89 \\ 2r=90 \\ r=45 \end{array}\right]$

$= \sum_{r=1}^{\cancel{89}\,45} 4r^2 - \sum_{r=1}^{\cancel{89}\,45} 4r + \sum_{r=1}^{\cancel{89}\,45} 1$

$= 125580 - 4\left(\frac{(45)(46)}{2}\right) + 45$

$= \cancel{121395}\ \ 121485$

Answer 121485

EXAMINER COMMENTARY

Part (a) is answered correctly.
In part (b) this student uses a different method from the one given in the mark scheme.
However, the method is correct and it uses the result of part (a), so it is awarded full marks.
Marks 2/2, 3/3

STUDENT B

6 (a) Find the value of

$$\sum_{r=1}^{45}(2r)^2$$

[2 marks]

$\sum_{r=1}^{45} 4r^2$

$= 4\sum_{r=1}^{45} r^2 = 4 \cdot \frac{1}{6} \times 45 \times (45+1) \times (45 \times 2+1)$

$= 125580$

Answer 125580

6 (b) Hence find the sum of the squares of all the odd numbers from 1 to 89

[3 marks]

$\sum_{r=1}^{89}(2r+1)^2 = \sum_{r=1}^{89}(4r^2+4r+1)$

$= 4\sum_{r=1}^{89} r^2 + 4\sum_{r=1}^{89} r + 1$

$= 4 \times \frac{1}{6} 89 \times (89+1) \times (2 \times 89+1) + 4 \times \frac{1}{2} \times 89 \times (89+1) + 1$

$= 971881$

Answer 971881

EXAMINER COMMENTARY

Part (a) is answered correctly.
In part (b) this student uses 89 as the upper limit of summation, which is an incorrect method.
As they have not used the result of part (a), they gain no marks.

COMPARISON

Because part (b) uses the word "hence", students can only gain marks if they use their result from part (a). This applies to any correct method, so the solution in script 3 can gain full marks even though it does not follow the mark scheme.

QUESTION

08

8 A hyperbola H_1 has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are positive constants.

H_1 intersects the x–axis at the points (6, 0) and (–6, 0)

The asymptotes of H_1 have equations

$$y = \frac{2}{3}x \quad \text{and} \quad y = -\frac{2}{3}x$$

8 (a) Find the values of a and b.

[2 marks]

$a =$

$b =$

8 (b) The hyperbola H_1 is translated by the vector $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to give the hyperbola H_2

8 (b) (i) Write down the equation of H_2

[1 mark]

Answer

8 (b) (iii) Find the equations of the tangents to H_2 which pass through the origin.

[5 marks]

Answer

8 (b) (ii) Show that, if the line $y = mx$ intersects H_2, then the x–coordinates of the points of intersection must satisfy the equation

$$(4 - 9m^2)\,x^2 - 32x - 80 = 0$$

[3 marks]

MARK SCHEME

Q	Answer	Mark	Total	Comments
8(a)	$a = 6$	B1	2	$a = \pm 6$ and $b = \pm 4$ SC1
	$b = 4$	B1		$a = 3$ and $b = 2$ SC1
8(b)(i)	$\frac{(x-4)^2}{36} - \frac{y^2}{16} = 1$	B1F	1	oe FT their a and b from 8(a)
8(b)(ii)	$\frac{(x-4)^2}{36} - \frac{m^2x^2}{16} = 1$	M1	3	
	$4(x-4)^2 - 9m^2x^2 = 144$	M1		
	$4(x^2 - 8x + 16) - 9m^2x^2 = 144$ $4x^2 - 32x + 64 - 9m^2x^2 = 144$			
	$(4 - 9m^2)x^2 - 32x - 80 = 0$	A1		for last line and either of the two preceding lines (oe)
8(b)(iii)	For equal roots $32^2 + 4(4 - 9m^2)(80) = 0$	M1	5	oe
	$1024 + 1280 - 2880m^2 = 0$	M1		
	$m^2 = \frac{4}{5}$	A1		
	$y = \frac{2\sqrt{5}}{5}x,\ y = -\frac{2\sqrt{5}}{5}x$	A1, A1		
Total	11			

STUDENT A

8 (a) Find the values of a and b.

[2 marks]

$\frac{36}{a^2} - \frac{0}{b^2} = 1$

$-\frac{b}{a} = -\frac{2}{3}$

$\frac{36}{a^2} = 1$

$-\frac{b}{6} = -\frac{2}{3}$

$a^2 = 36$

$a = \pm 6$

$b = 4$

$a =$ 6

$b =$ 4

8 (b) The hyperbola H_1 is translated by the vector $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to give the hyperbola H_2

8 (b) (i) Write down the equation of H_2

[1 mark]

$\frac{(x-4)^2}{36} - \frac{y^2}{16} = 1$

Answer $\frac{(x-4)^2}{36} - \frac{y^2}{16} = 1$

8 (b) (ii) Show that, if the line $y = mx$ intersects H_2, then the x-coordinates of the points of intersection must satisfy the equation

$$(4 - 9m^2)x^2 - 32x - 80 = 0$$

[3 marks]

$\frac{(x^2 - 8x + 16)}{36} - \frac{(mx)^2}{16} = 1$

$\frac{x^2 - 8x + 16}{36} - \frac{m^2x^2}{16} = 1$

$16(x^2 - 8x + 16) - 36m^2x^2 = 576$

$16x^2 - 128x + 256 - 36m^2x^2 = 576$

$4x^2 - 32x - 80 - 9m^2x^2 = 0$

$(4 - 9m^2)x^2 - 32x - 80 = 0$

8 (b) (iii) Find the equations of the tangents to H_2 which pass through the origin.

[5 marks]

$b^2-4ac=0 \qquad (4-9m^2)x^2-32x-80=0$

$(-32)^2-4(4-9m^2)(-80)=0$

$1024+320(4-9m^2)=0$

$1024+1280-2880m^2=0$

$2304=2880m^2$

$\frac{4}{5}=m^2$

$m=\pm\sqrt{\frac{4}{5}}=\pm\frac{2\sqrt{5}}{5}$

$m=\frac{2\sqrt{5}}{5} \qquad m=-\frac{2\sqrt{5}}{5}$

$-\frac{16}{5}x^2-32x-80=0$

$(x+5)^2=0$

$x=-5 \qquad +c=0$

$y=\left(-\frac{2\sqrt{5}}{5}\right)x \qquad y=\left(\frac{2\sqrt{5}}{5}\right)x$

Answer $y=\left(-\frac{2\sqrt{5}}{5}\right)x$ and $y=\left(\frac{2\sqrt{5}}{5}\right)x$

EXAMINER COMMENTARY

This solution proceeds logically and uses all the information given. In particular the fact that H_1 intersects the x-axis at (6,0) and (-6, 0) is used to deduce that $a=6$

Marks 2/2, 1/1, 3/3, 5/5

STUDENT B

8 (a) Find the values of a and b.

[2 marks]

$\text{Asy} = \pm \frac{b}{a}x \quad \therefore b=2 \quad a=3$

$a =$ 3

$b =$ 2

8 (b) The hyperbola H_1 is translated by the vector $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to give the hyperbola H_2

8 (b) (i) Write down the equation of H_2

[1 mark]

$$\frac{(x-4)^2}{9} - \frac{y^2}{4} = 1$$

Answer $\frac{(x-4)^2}{9} - \frac{y^2}{4} = 1$

8 (b) (ii) Show that, if the line $y = mx$ intersects H_2, then the x-coordinates of the points of intersection must satisfy the equation

$$(4 - 9m^2)x^2 - 32x - 80 = 0$$

[3 marks]

$$\frac{x^2 - 8x + 16}{9} - \frac{m^2x^2}{4} = 1$$

$$4x^2 - 32x + 64 - 9m^2x^2 = 36$$

$$(4 - 9m^3)x^2 - 32x - 80 = 0$$

8 (b) (iii) Find the equations of the tangents to H_2 which pass through the origin.

[5 marks]

$(4-9m^2)x^2 - 32x - 80 = 0$

$b^2 - 4ac = 0$

$1024 - 4(4-9m^2)(-80) = 0$

$1024 \cancel{+16} \cancel{+36m^2} + 1280 - 2880m^2 = 0$

$2304 \cancel{-256} = 2880m^2$

$\frac{4}{5} = m^2$

$m = \pm \frac{2\sqrt{5}}{5}$

$y = \frac{2\sqrt{5}}{5}x \qquad y = -\frac{2\sqrt{5}}{5}x$

Answer $y = \pm \frac{2\sqrt{5}}{5}x$

EXAMINER COMMENTARY

This student gains 1 mark in part (a) for the incorrect result $a = 3$ and $b = 2$
This error occurred because they did not use the fact that H_1 intersects the x-axis at (6,0) and (-6, 0)
They are then awarded a follow-through mark in part (b)(i), and two method marks in part (b)(ii). It is not possible for them to gain full marks for part (b)(ii), as the result to be proved is not consistent with their equation for H_2
They are able to proceed from the equation given in part (b)(ii) to answer part (b)(iii) correctly.
Marks 1/2, 1/1, 2/3, 5/5

COMPARISON

Despite an early error, the solution in script 2 recovers to gain most of the marks for this question. This error could have been avoided by ensuring that all the given information was taken into account.

QUESTION

09

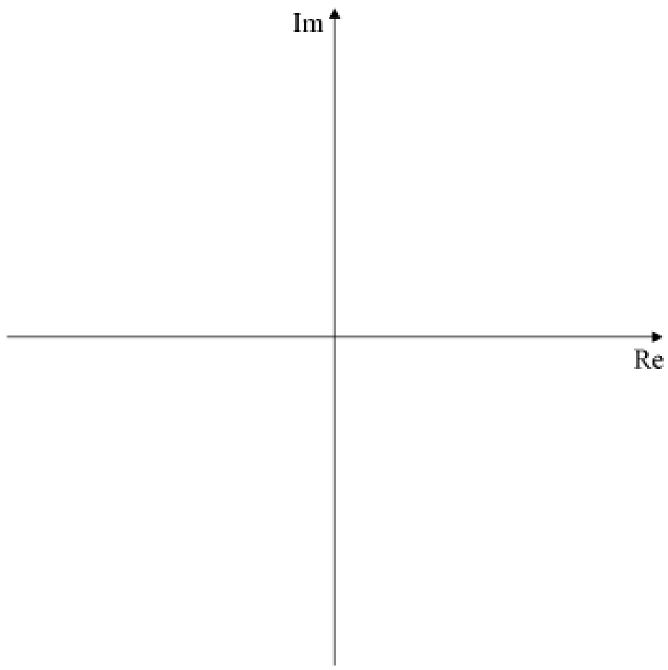
9 The locus L of points satisfies the equation $|z| = |z - 4 - 4\text{i}|$

The point P represents the complex number $4 + 4\text{i}$

The circle C has centre P and touches L.

9 (a) Sketch L and C on the same Argand diagram in the space below.

[4 marks]



9 (b) Given that z_1 lies on C, find the maximum possible value of $|z_1|$

[3 marks]

Answer

9 (c) Given that z_2 lies on C, find the minimum possible value of $\arg(z_2)$

[3 marks]

Answer

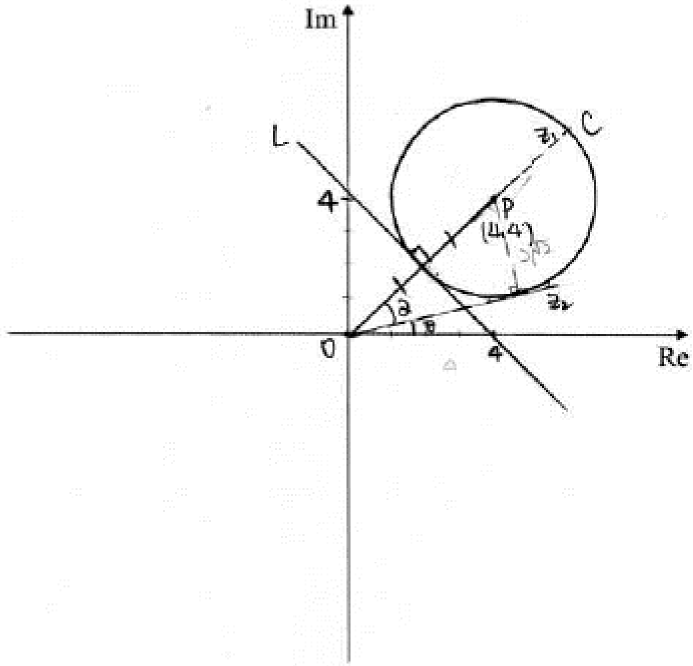
MARK SCHEME

Q	Answer	Mark	Total	Comments
9(a)	Line in 1st quadrant with negative gradient	B1	**4**	
	Crosses axes at 4 and $4i$ **and** extends into quadrants 2 and 4	B1		Or clearly shows the perpendicular bisector of the line connecting 0 and $4 + 4i$
	Circle centre $4 + 4i$	B1		
	Touches their L	B1		
9(b)	$r = 2\sqrt{2}$ or furthest point = 6 + 6i	B1	3	
	$\lvert z\rvert_{max} = OP + r$ or $\lvert 6 + 6i\rvert$	M1		PI
	$6\sqrt{2}$	A1		
9(c)	Radius perpendicular to tangent	M1	3	
	Angle $TOP = 30°$ or $\pi/6$	A1		
	15° or $\pi/12$	A1		
Total	**10**			

STUDENT A

9 (a) Sketch L and C on the same Argand diagram in the space below.

[4 marks]



9 (b) Given that z_1 lies on C, find the maximum possible value of $|z_1|$

[3 marks]

$|z_1|$ max $= 4\sqrt{2} + 2\sqrt{2}$ (← radius ~~..~~ + origin length)

$= 6\sqrt{2}$

$\left(OP = \sqrt{4^2+4^2} = 4\sqrt{2} \right)$

radius $= 2\sqrt{2}$

Answer ~~$6\sqrt{2}$~~ $6\sqrt{2}$

9 (c) Given that z_2 lies on C, find the minimum possible value of $\arg(z_2)$

[3 marks]

$\arg z_1 = 45° - \alpha$

$\sin\alpha = \dfrac{2\sqrt{2}}{4\sqrt{2}}$ $\alpha = 30°$

$\therefore \arg z_2 = 45° - 30°$

$= 15°$

Answer $15°$

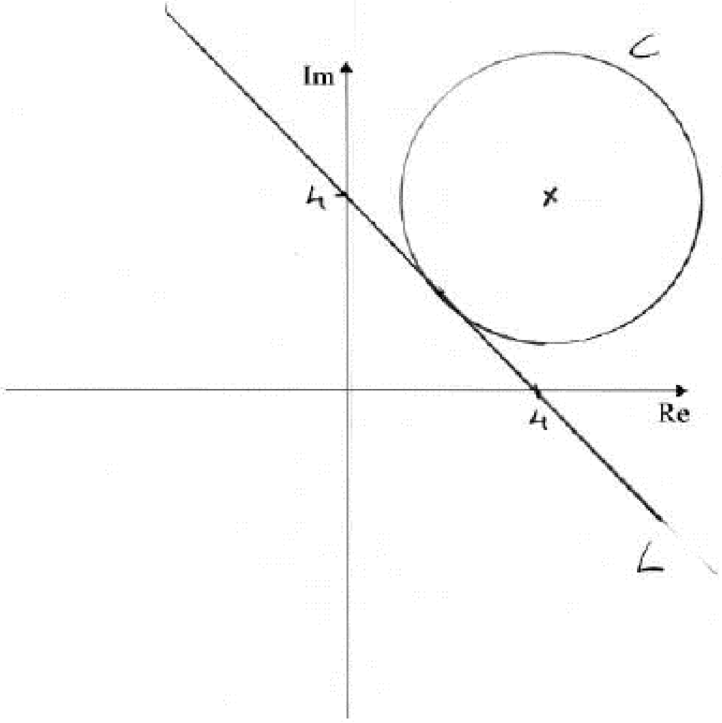
EXAMINER COMMENTARY

In part (a) the diagram clearly shows the centre of the circle and the axis intercepts.
The solutions to parts (b) and (c) are also shown on the diagram. This is not required in order to gain the marks for parts (b) and (c), but it is recommended as it can help the student to see the overall picture.
In part (c) the student refers to an angle α; the diagram makes it clear which angle this is. The solution of each part is clear, logical and easy to follow.
Marks 4/4, 3/3, 3/3

STUDENT B

9 (a) Sketch L and C on the same Argand diagram in the space below.

[4 marks]



9 (b) Given that z_1 lies on C, find the maximum possible value of $|z_1|$

[3 marks]

~~Line to L ⊥ −x (Passes through~~

$y = x + c$

$4 = 4 + c$

$c = 0$ ∴ Equ^n $y = x$

$y = x$, $y = -x + 4$ ∴ $x = -x + 4 \Rightarrow 2x = 4$ ∴ $x = 2$

∴ $x = 2$, $y = 2$ ∴ Intersect at $2 + 2i$

$|2+2i| = r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

∴ Max dist. from origin $= 4 + 4i + 2\sqrt{2}i$

$|4 + 4i + 2\sqrt{2}i| = \sqrt{4^2 + (4 + 2\sqrt{2})^2} = \sqrt{40 + 16\sqrt{2}}$

Answer ~~$4 + 2i(2 + \sqrt{2})$~~ $\sqrt{40 + 16\sqrt{2}}$

9 (c) Given that z_2 lies on C, find the minimum possible value of $\arg(z_2)$

[3 marks]

z_2 is at bottom of $C = 4 + 4i - 2\sqrt{2}i$

$\therefore \arg z_2 = \tan^{-1}\left(\frac{4-2\sqrt{2}}{4}\right) = 0.285$

Answer 0.285

EXAMINER COMMENTARY

In part (a) the centre of the circle is marked with a cross but it is not labelled with the letter P, the complex number 4 = 4i or the Cartesian coordinates (4, 4). Any one of these would be sufficient to gain the fourth mark.

In part (b) the student takes an algebraic approach, which enables them to find the point where the line meets the circle, and the distance of this point from the origin. However, after this they take a wrong turn, and their solution does not gain any marks after the first one.

In part (c) they incorrectly assume that the point represented by z_2 is directly below P. If they had drawn a tangent from the origin to the circle they might have seen that z_2 is the point where the tangent meets the circle.

Marks 3/4, 1/3, 0/3

COMPARISON

In this type of question it helps to be flexible in switching between algebraic and geometric approaches. Students who tackle a question algebraically are advised to use the geometric properties of the diagram as a check that their solution is correct. This could help them to avoid the type of error seen in script 3.

QUESTION

10

10 A curve C has the equation

$$y = \frac{(x+5)(x+1)}{x(x-4)}$$

10 (a) State the equations of the asymptotes of C.

[3 marks]

Answer

10 (b) The line $y = k$ intersects the curve C.

10 (b) (i) Show that

$$4k^2 + 17k + 4 \geq 0$$

[4 marks]

10 (b) (ii) **Hence** find the coordinates of the stationary points of the curve C.

No credit will be given for solutions using differentiation.

[5 marks]

MARK SCHEME

Q	Answer	Mark	Total	Comments
10(a)	$y = 1$	B1	3	
	$x = 0$	B1		
	$x = 4$	B1		
10(b)(i)	$k(x^2 - 4x) = x^2 + 6x + 5$	M1	4	
	$(k-1)x^2 - (4k+6)x - 5 = 0$	A1		
	$(4k+6)^2 + 20(k-1) \geq 0$	M1		
	$16k^2 + 68k + 16 \geq 0$ $4k^2 + 17k + 4 \geq 0$	A1		for both lines
10(b)(ii)	Equal roots: $4k^2 + 17k + 4 = 0$	M1	5	
	$k = -4$ Or $-\frac{1}{4}$	A1		
	Substituting at least one value of k into $(k-1)x^2 - (4k+6)x - 5 = 0$	M1		
	$(1, -4)$ or $\left(-2, -\frac{1}{4}\right)$	A1		
	$(1, -4)$ and $\left(-2, -\frac{1}{4}\right)$ and no extras	A1		
Total	12			

STUDENT A

10 (a) State the equations of the asymptotes of C.

[3 marks]

$x=0$. $x=4$. $y=1$

when $x \to \infty$

$RHS = \frac{\infty \times \infty}{\infty \times \infty} = 1$

Answer $x=0$, $x=4$, $y=1$

10 (b) The line $y = k$ intersects the curve C.

10 (b) (i) Show that

$$4k^2 + 17k + 4 \geq 0$$

[4 marks]

$x(x-4) \cdot k = x^2+6x+5$.

$kx^2 - 4kx = x^2+6x+5$.

$(k-1)x^2 - (4k+6)x - 5 = 0$.

$\because \Delta \geq 0$.

$(4k+6)^2 + 4 \times 5 \times (k-1) \geq 0$.

$16k^2+48k+36 + 20k - 20 \geq 0$.

$16k^2+68k + 16 \geq 0$.

$4k^2+17k+4 \geq 0$.

10 (b) (ii) **Hence** find the coordinates of the stationary points of the curve C.

No credit will be given for solutions using differentiation.

[5 marks]

$4k^2+17k+4=0$

$(4k+1)(k+4)=0$

$k=-\frac{1}{4}$ or $k=-4$

$(-\frac{1}{4}-1)x^2-(4\times\frac{-1}{4}+6)x-5=0$ $\quad$ $(-4-1)x^2-(4\times-4+6)x-5=0$

$-\frac{5}{4}x^2-5x-5=0$ $\quad$ $-5x^2+10x-5=0$

$-5x^2-20x-20=0$ $\quad$ $x^2-2x+1=0$

$x^2+4x+4=0$ $\quad$ $(x-1)^2=0$

$x=-2$ $\quad$ $x=1$

Answer $(-2,-\frac{1}{4})$ $\quad$ $(1,-4)$.

EXAMINER COMMENTARY

In part (a) some of the working shown, where infinity is used within a fraction, is not strictly correct. However, as the answer is correct, the student is not penalised.
Part (b)(i) and part (b) (ii) are both answered clearly and correctly.
Marks 3/3, 4/4, 5/5

STUDENT B

10 (a) State the equations of the asymptotes of C.

[3 marks]

$y = \cancel{\dfrac{x+6+\frac{5}{x}}{x-4}}$

$x = 4$

$x = 0$

$y = \dfrac{-6}{4}$

$y = \dfrac{x^2+6x+5}{x^2-4x} \quad \div x$

$y = \dfrac{x+6+\frac{5}{x}}{x-4} \quad x \to \infty \quad \dfrac{\infty + 6}{\infty + 4}$

$= \dfrac{6}{4}$

Answer $x = 4 \quad x = 0 \quad y = \dfrac{-6}{4}$

10 (b) The line $y = k$ intersects the curve C.

10 (b) (i) Show that

$$4k^2 + 17k + 4 \geq 0$$

[4 marks]

$K = \dfrac{x^2+6x+5}{x^2-4x}$

$Kx^2 - 4Kx = x^2 + 6x + 5$

$(K-1)x^2 + (-4K-6)x - 5 = 0$

$b^2 - 4ac \geq 0$

$16K^2 + 48K + 36 - 4(K-1)(-5) \geq 0$

$16K^2 + 48K + 36 + 20K - 20 \geq 0$

$16K^2 + 68K + 16 \geq 0$

$4K^2 + 17K + 4 \geq 0$

10 (b) (ii) **Hence** find the coordinates of the stationary points of the curve C.

No credit will be given for solutions using differentiation.

[5 marks]

$4k^2+17k+4=0$

$k=-\frac{1}{4}$ $\quad k=-4$

$y=-4$ $\quad y=-\frac{1}{4}$

$y=\frac{x^2+6x+5}{x^2-4x} \longrightarrow y=-4 \quad x=-\frac{3}{32}=-0.0938$

$y=-\frac{1}{4} \quad x=\frac{57}{17}=3.353$

Answer $\left(\frac{-3}{32}, -4\right)$ and $\left(\frac{57}{17}, \frac{-1}{4}\right)$

EXAMINER COMMENTARY

In part (a) the two vertical asymptotes are correct, but the student concentrates on the coefficients of x rather than x^2 in their attempt to find the horizontal asymptote, leading to the wrong answer.
Part (b) (i) is answered correctly.
In part (b)(ii) the student successfully finds the required values of k, that is, the y-coordinates of the stationary points. They appear to be about to equate each of these in turn to the original equation for y. This is possibly not the quickest method, but it is certainly correct and would lead to the right answer. However, no equation is actually written down so there are no marks available beyond the first two.
Marks 2/3, 4/4/ 2/5

COMPARISON

Both students have scored well on this question. The student in script 2 could possibly have made more progress in part (b)(ii) by being more methodical and writing down an equation as their next step.

FURTHER GUIDANCE AND CONTACTS

You can contact the subject team directly at **maths@oxfordaqaexams.org.uk**

Please note: We aim to respond to all email enquiries within two working days.

Our UK office hours are Monday to Friday, 8am – 5pm local time.

OXFORD
INTERNATIONAL
AQA EXAMINATIONS

OXFORD INTERNATIONAL AQA EXAMINATIONS
GREAT CLARENDON STREET, OXFORD, OX2 6DP
UNITED KINGDOM

enquiries@oxfordaqaexams.org.uk
oxfordaqaexams.org.uk