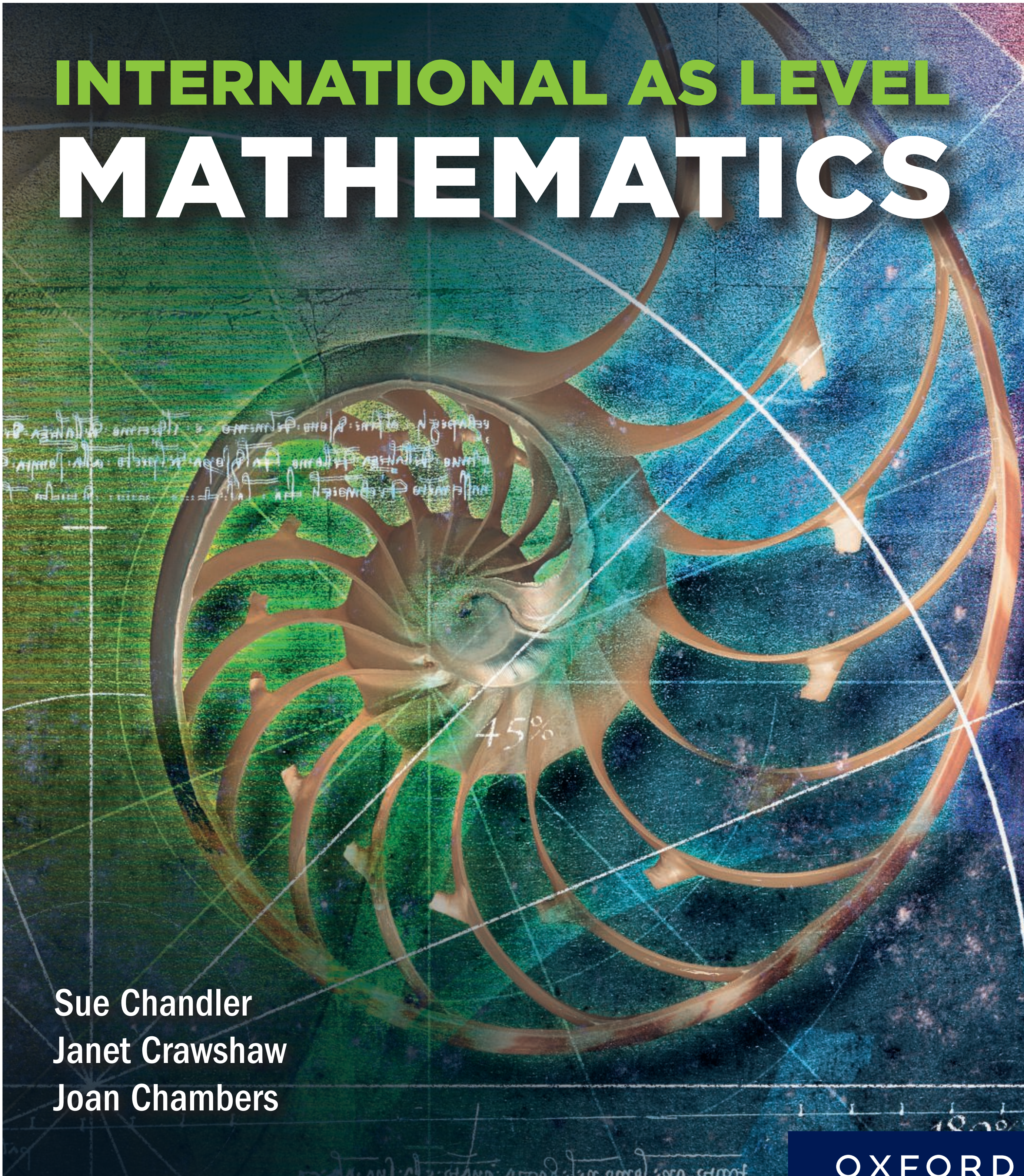
Published for
OXFORD INTERNATIONAL AQA EXAMINATIONS

INTERNATIONAL AS LEVEL
MATHEMATICS



Sue Chandler
Janet Crawshaw
Joan Chambers

OXFORD

Contents

1 Expanding Brackets, Surds and Indices

Introduction

Working with algebraic expressions is needed in any mathematics course beyond GCSE. This chapter gives the facts and the practice necessary for you to develop these skills.

Objectives

By the end of this chapter, you should know how to...

- Identify like and unlike terms.
- Expand brackets.
- Explain the meaning of a surd.
- Simplify an expression containing surds.
- Work with numbers in index form.

Recap

You need to remember...

- An algebraic expression does not contain an equals sign but an equation does contain an equals sign.
  For example, $3x^2 - 4$ is an expression, $2x + 6y = 7$ is an equation.
- If a string of numbers and letters are multiplied, the multiplication can be done in any order. For example,
  $2p \times 3q = 2 \times p \times 3 \times q = 2 \times 3 \times p \times q = 6pq$
- $x^2 = x \times x$, $x^5 = x \times x \times x \times x \times x$, etc.

1.1 Algebraic expressions

The **terms** in an algebraic expression are the parts separated by a plus or minus sign.

Like terms contain the same letters to the same powers; like terms can be added or subtracted.

For example, $2ab$ and $5ab$ are like terms and can be added,

so $\quad 2ab + 5ab = 7ab$

Unlike terms contain different letters; they cannot be added or subtracted.

For example, ab and ac are unlike terms because they contain different letters. Also x^2 and x^3 cannot be added because they are to different powers.

Example 1

Question

Simplify $5x - 3(4 - x)$

Answer

$$5x - 3(4 - x) = 5x - 12 + 3x$$
$$= 8x - 12$$

Note

Remember that $-3(4 - x)$ means 'multiply every term inside the bracket by minus 3. Remember that $(-3) \times (-x) = +3x$.

Coefficients

We can identify a term in an expression by using the letter, or combination of letters, involved.

For example $2x^2$ is 'the term in x^2',

$3xy$ is 'the term in xy'.

The number (including its sign) in front of the letters is called the **coefficient**.

For example in the term $2x^2$, 2 is the coefficient of x^2

in the term $3xy$, 3 is the coefficient of xy.

If no number is written in front of a term, the coefficient is 1 or -1, depending on the sign of the term.

For example, in the expression $x^3 + 5x^2y - y^3 + 2$

the coefficient of x^3 is 1

the coefficient of x^2y is 5

the coefficient of y^3 is -1.

The term 2 has no variable. This is called a constant term.

Exercise 1

Simplify.

1 $2x^2 - 4x + x^2$
2 $5a - 4(a+3)$
3 $2y - y(x-y)$
4 $8pq - 9p^2 - 3pq$
5 $4xy - y(x-y)$
6 $x^3 - 2x^2 + x^2 - 4x + 5x + 7$
7 $t^2 - 4t + 3 - 2t^2 + 5t + 2$
8 $2(a^2 - b) - a(a+b)$
9 $3 - (x-4)$
10 $5x - 2 - (x+7)$
11 $3x(x+2) + 4(3x-5)$
12 $a(b-c) - c(a-b)$
13 $2ct(3-t) + 5t(c-11t)$
14 $x^2(x+7) - 3x^3 + x(x^2-7)$
15 $(3y^2 + 4y - 2) - (7y^2 - 20y + 8)$
16 Write down the coefficient of x in $x^2 - 7x + 4$.
17 What is the coefficient of xy^2 in the expression $y^3 + 2xy^2 - 7xy$?
18 For the expression $x^3 - 3x + 7$, write down the coefficient of

a x^3 **b** x^2 **c** x

Note

$-(x-4)$ means $-1(x-4)$.

1.2 Expansion of Two Brackets

Expanding brackets means multiplying them out to remove the brackets.

To expand $(2x+4)(x-3)$ each term in the first bracket is multiplied by each term in the second bracket.

$$(2x+4)(x-3) = 2x^2 - 6x + 4x - 12$$

$$= 2x^2 - 2x - 12$$

Exercise 2

Expand and simplify.

1 $(x+2)(x+4)$
2 $(x+5)(x+3)$
3 $(a+6)(a+7)$
4 $(t+8)(t+7)$
5 $(s+6)(s+11)$
6 $(2x+1)(x+5)$
7 $(5y+3)(y+5)$
8 $(2a+3)(3a+4)$
9 $(7t+6)(5t+8)$
10 $(11s+3)(9s+2)$
11 $(x-3)(x-2)$
12 $(y-4)(y-1)$
13 $(a-3)(a-8)$
14 $(b-8)(b-9)$
15 $(p-3)(p-12)$
16 $(2y-3)(y-5)$
17 $(x-4)(3x-1)$
18 $(2r-7)(3r-2)$
19 $(4x-3)(5x-1)$
20 $(2a-b)(3a-2b)$
21 $(x-3)(x+2)$
22 $(a-7)(a+8)$
23 $(y+9)(y-7)$
24 $(s-5)(s+6)$
25 $(q-5)(q+13)$
26 $(2t-5)(t+4)$
27 $(x+3)(4x-1)$
28 $(2q+3)(3q-5)$
29 $(x+y)(x-2y)$
30 $(s+2t)(2s-3t)$

Difference of two squares

The expansion of $(x-4)(x+4)$ is a special case.

$$\begin{aligned}(x-4)(x+4) &= x^2-4x+4x-16\\ &= x^2-16\end{aligned}$$

Any expansion of the form $(x+b)(x-b)=x^2-b^2$ is known as the difference of two squares.

Squares

$(2x+3)^2$ means $(2x+3)(2x+3)$

$$\begin{aligned}\therefore\quad (2x+3)^2 &= (2x+3)(2x+3)\\ &= (2x)^2+(2)(2x)(3)+(3)^2\\ &= 4x^2+12x+9\end{aligned}$$

In general, $(ax+b)^2 = a^2x^2+(2)(ax)(b)+b^2$

$$= a^2x^2+2abx+b^2$$

and $(ax-b)^2 = a^2x^2-2abx+b^2$

Exercise 3

Expand and simplify.

1 $(x-2)(x+2)$
2 $(5+x)(5-x)$
3 $(x+3)(x-3)$
4 $(2x-1)(2x+1)$
5 $(x-8)(x+8)$
6 $(x-a)(x+a)$
7 $(x-1)(x+1)$
8 $(3b+4)(3b-4)$
9 $(2y-3)(2y+3)$
10 $(ab+6)(ab-6)$
11 $(5x+1)(5x-1)$
12 $(xy+4)(xy-4)$

Note

Questions 1 to 6 show that the expansion of $(ax+b)(ax-b)$ can be written down directly, so $(ax+b)(ax-b)=a^2x^2-b^2$. Use this result to expand the brackets in Questions 7 to 12.

Expand.

13 $(x+4)^2$
14 $(x+2)^2$
15 $(2x+1)^2$
16 $(3x+5)^2$
17 $(2x+7)^2$
18 $(x-1)^2$
19 $(x-3)^2$
20 $(2x-1)^2$
21 $(4x-3)^2$
22 $(5x-2)^2$
23 $(3t-7)^2$
24 $(x+y)^2$
25 $(2q+9)^2$
26 $(3q-11)^2$
27 $(2x-5y)^2$
28 Expand and simplify $(x-2)^2(3x-4)$. Write down the coefficients of x^2 and x.

Important expansions

These general results should be memorised.

$(ax+b)^2 = a^2x^2 + 2abx + b^2$

$(ax-b)^2 = a^2x^2 - 2abx + b^2$

$(ax+b)(ax-b) = a^2x^2 - b^2$

The next exercise has different expansions including some given above.

Example 2

Question

Expand $(4p+5)(3-2p)$

Answer

$$\begin{aligned}(4p+5)(3-2p) &= (5+4p)(3-2p)\\ &= 15-10p+12p-8p^2\\ &= 15+2p-8p^2\end{aligned}$$

Harder expansions

Expanding expressions such as $(x-2)(x^2-x+5)$ should be done systematically.

First multiply each term of the second bracket by x, writing down the separate results as they are found. Then multiply each term of the second bracket by -2. Do not collect like terms at this stage.

$$(x-2)(x^2-x+5)$$
$$= x^3 - x^2 + 5x - 2x^2 + 2x - 10$$

Now collect like terms

$$= x^3 - 3x^2 + 7x - 10$$

Example 3

Question

Expand $(x+2)(2x-1)(x+4)$

Answer

$$\begin{aligned}(x+2)(2x-1)(x+4) &= (x+2)(2x^2+7x-4)\\ &= 2x^3+7x^2-4x+4x^2+14x-8\\ &= 2x^3+11x^2+10x-8\end{aligned}$$

Note

First expand the last two brackets.

Exercise 4

Expand.

1. $(2x-3)(4-x)$
2. $(x-7)(x+7)$
3. $(6-x)(1-4x)$
4. $(7p+2)(2p-1)$
5. $(3p-1)^2$
6. $(5t+2)(3t-1)$
7. $(4-p)^2$
8. $(4t-1)(3-2t)$
9. $(x+2y)^2$
10. $(4x-3)(4x+3)$
11. $(3x+7)^2$
12. $(R+3)(5-2R)$
13. $(a-3b)^2$
14. $(2x-5)^2$
15. $(7a+2b)(7a-2b)$
16. $(3a+5b)^2$

Expand and simplify

17 $(x-2)(x^2+x+1)$

18 $(3x-2)(x^2-x-1)$

19 $(2x-1)(2x^2-3x+5)$

20 $(x-1)(x^2-x-1)$

21 $(2x+3)(x^2-6x-3)$

22 $(x+1)(x+2)(x+3)$

23 $(x+4)(x-1)(x+1)$

24 $(x-2)(x-3)(x+1)$

25 $(x+1)(2x+1)(x+2)$

26 $(x+2)(x+1)^2$

27 $(2x-1)^2(x+2)$

28 $(3x-1)^3$

29 $(4x+3)(x+1)(x-4)$

30 $(x-1)(2x-1)(2x+1)$

31 $(2x+1)(x+2)(3x-1)$

32 $(x+1)^3$

33 $(x-2)(x+2)(x+1)$

34 $(x+3)(2x+3)(x-1)$

35 $(3x-2)(2x+5)(4x-1)$

36 $2(x-7)(2x+3)(x-5)$

37 $(x+y)^3$

38 $(x+y)^4$

39 $(x^2-5)^3$

40 $(2-3x^2)^3$

41 Find the coefficients of x^3 and x^2 in the expansion of $(x-4)(2x+3)(3x-1)$.

42 State the coefficient of x^6 in the expansion of $(2x^3-3)^3$.

Note

The result from question 37 should be memorized:
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

Note

For question 42 use the general result of the expansion of $(a+b)^3$ and replace a by $2x^3$ and b by -3.

1.3 Square roots and other roots

When a number is given as the product of two equal factors, that factor is called the **square root** of the number, for example

$4 = 2\times 2 \Rightarrow 2$ is the square root of 4.

This is written $2=\sqrt{4}$.

-2 is also a square root of 4, as $4=-2\times -2$ but $\sqrt{4}\neq -2$.

The symbol $\sqrt{}$ is used *only for the positive square root*.

So, although $x^2=4\Rightarrow x=\pm 2$, the only value of $\sqrt{4}$ is 2. The negative square root of 4 is written $-\sqrt{4}$.

When both square roots are wanted, we write $\pm\sqrt{4}$.

Note

The symbol $\Rightarrow$ means gives or implies.

Cube roots

When a number is given as the product of three equal factors, that factor is called the **cube root** of the number.

For example $27 = 3\times 3\times 3$ so 3 is the cube root of 27.

This is written $\sqrt[3]{27}=3$.

Other roots

The notation used for square and cube roots can be extended to represent fourth roots, fifth roots, and so on.

For example $16 = 2\times 2\times 2\times 2 \Rightarrow \sqrt[4]{16} = 2$

and $243 = 3\times 3\times 3\times 3\times 3 \Rightarrow \sqrt[5]{243} = 3$

In general, if a number, n, can be written as the product of p equal factors then each factor is called the p^{th} root of n and is written $\sqrt[p]{n}$.

Summary

Expansions

$(ax+b)^2 = a^2x^2 + 2abx + b^2$

$(ax-b)^2 = a^2x^2 - 2abx + b^2$

$(ax+b)(ax-b) = a^2x^2 - b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Surds

The denominator of $\frac{a}{\sqrt{b}}$ can be rationalized by multiplying numerator and denominator by $\sqrt{b}$.

The denominator of $\frac{a}{b+\sqrt{c}}$ can be rationalized by multiplying numerator and denominator by $b-\sqrt{c}$.

Indices

$a^n \times a^m = a^{n+m}$

$a^n \div a^m = a^{n-m}$

$(a^n)^m = a^{nm}$

$\sqrt[n]{a} = a^{\frac{1}{n}}$

$a^0 = 1$

Review

1. Simplify $a^2(3-a)-(a-a^3)$.
2. Expand and simplify $(2x-7)(x+5)$.
3. Expand and simplify $(4x-3)^2$.
4. Expand and simplify $(2x-3)(2x+3)$.
5. Expand and simplify $(3-5x)(2x+1)$.
6. Expand and simplify $(2x-3y)(x+y)^2$.
7. State the coefficient of a^2b in the expansion of $(3a-2b)^3$.
8. Express $\sqrt{150}$ in terms of the simplest possible surd.
9. Expand $\left(4-3\sqrt{3}\right)^2$ and simplify if possible.

For Questions 10, 11and 12 state the letter that gives the correct answer.

10. $\frac{1-\sqrt{2}}{1+\sqrt{2}}$ is equal to

 a 1 **b** -1 **c** $3-2\sqrt{2}$ **d** $2\sqrt{2}-3$

11. $\frac{p^{-\frac{1}{2}} \times p^{\frac{3}{4}}}{p^{-\frac{1}{4}}}$ simplifies to

 a $p^{\frac{1}{2}}$ **b** $p^{-\frac{1}{2}}$ **c** $p^{\frac{3}{4}}$ **d** p

12 $\dfrac{5^{\frac{1}{4}} \times 5 \times 5^{\frac{1}{6}}}{\sqrt{5}} = 5^p$. The value of p is

a $-\dfrac{1}{12}$ **b** $\dfrac{11}{12}$ **c** $1\dfrac{11}{12}$

Assessment

1 **a** Given that $\dfrac{1}{27} = 3^r$ state the value of r.

b Given that $\sqrt{3} = 3^r$ state the value of r.

2 **a** The expression $(x-3)(x^2-5x+6)$ can be written in the form $x^3+px^2+qx-18$. Show that $p=-8$ and find the value of q.

b The expression $(x^2+6)^3$ can be written in the form $x^6+px^4+qx^2+216$. Find the values of p and q.

3 **a** Show that $\sqrt{72} = p\sqrt{2}$ giving the value of p.

b Show that $\dfrac{\sqrt{8}+\sqrt{18}}{\sqrt{2}} = n$ where n is an integer. State the value of n.

c Show that $\dfrac{2\sqrt{2}-1}{2-\sqrt{2}}$ can be expressed in the form $p+q\sqrt{2}$ where p and q are rational numbers. State the values of p and q.

4 **a** Express $\dfrac{2^{\frac{1}{2}} \times 2^{-\frac{1}{4}}}{2^{\frac{3}{4}}}$ as 2^r. State the value of r.

b Express $\left(\dfrac{36x^2}{16}\right)^{\frac{1}{2}}$ in the form ax^b giving the values of a and b. Give the value of a as a fraction its simplest form.

5 **a** Expand $(x-1)^2(2x+3)$.

b Find the coefficients of x^2 and x in the expansion of $(x-2)(2x+3)(x+2)$.

6 **a** **i** Simplify $\left(3\sqrt{2}\right)^2$

ii Show that $\left(3\sqrt{2}-2\right)^2 + \left(3+2\sqrt{2}\right)^2$ is an integer and find its value.

b Express $\dfrac{4\sqrt{5}-7\sqrt{2}}{2\sqrt{5}+\sqrt{2}}$ in the form $m-\sqrt{n}$, where m and n are integers.

AQA MPC1 January 2012

7 A rectangle has length $\left(9+5\sqrt{3}\right)$ cm and area $\left(15+7\sqrt{3}\right)$ cm^2.

Find the width of the rectangle, giving your answer in the form $\left(m+n\sqrt{3}\right)$ cm, where m and n are integers.

AQA MPC1 June 2014