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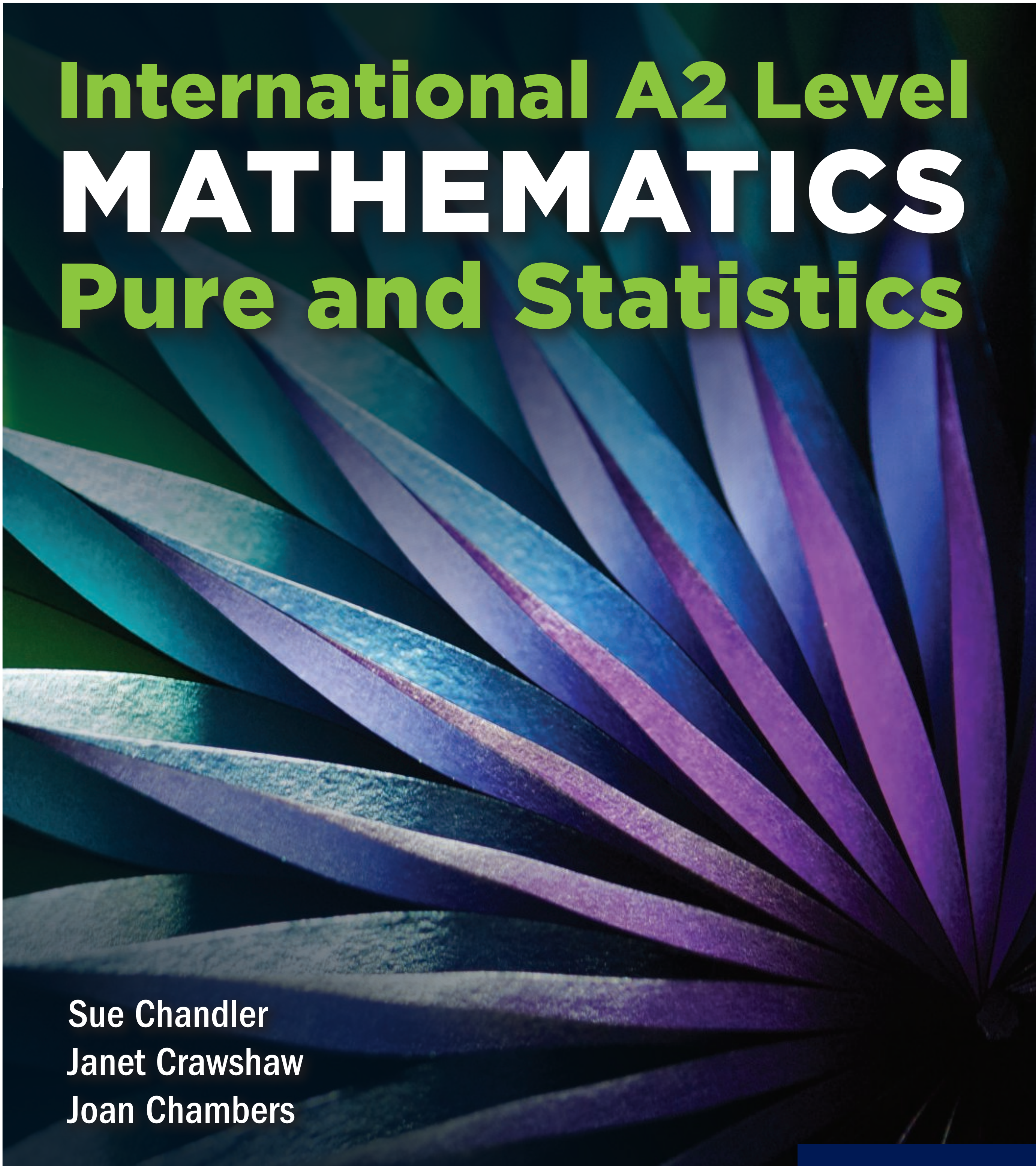
Pure and Statistics



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A2 Pure

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1 Functions

Introduction

This chapter extends the work on functions introduced at AS-level and gives various methods for expressing algebraic fractions in simpler forms. These methods are needed later in the course for integrating and differentiating fractions.

Objectives

By the end of this chapter, you should know how to...

- Define a function, range of a function and domain of a function.
- Introduce inverse functions, composite functions and modulus functions.
- Use combinations of transformations to help to sketch graphs.
- Simplify an algebraic fraction by dividing by common factors.
- Decompose algebraic fractions into simpler fractions.

Recap

You will need to remember...

- The properties and the shapes of the graphs of linear, quadratic, exponential and trigonometric functions.
- The effect of simple transformations on a graph, including translations, one-way stretches and reflections in the x- and y-axes.
- The Cartesian equation of a curve gives the relationship between the x- and y-coordinates of points on the curve.
- How to complete the square for a quadratic function.
- How to factorise quadratic expressions.
- The remainder theorem.

1.1 Functions

When you substitute any number for x in the expression $x^2 - 2x$, you get a single answer.

For example when $x = 3$, $x^2 - 2x = 3$.

However, when you substitute a positive number for x in the expression $\pm\sqrt{x}$, you have two possible answers.

For example when $x = 4$, $\pm\sqrt{x} = -2$ or 2.

A *function* of one variable is such that when a number is substituted for the variable, there is only one answer.

Therefore $x^2 - 2x$ is an example of a function f and can be written as $f(x) = x^2 - 2x$.

However, $\pm\sqrt{x}$ is not a function of x because any positive value of x gives two answers.

Domain and range

The set of values which the variable in a function can take is called the *domain* of the function.

The domain does not have to contain all possible values of the variable; it can be as wide, or as restricted, as needed. Therefore to define a function fully, the domain must be stated.

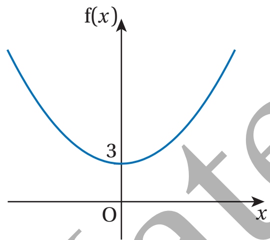
If the domain is not stated, assume that it is the set of all **real numbers** (the set of real numbers is denoted by $\mathbb{R}$).

For each domain, there is a corresponding set of values of f(x). These are values which the function can take for values of x in that particular domain. This set is called the *range* of the function.

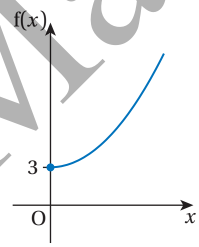
Look at the expression $x^2 + 3$.

A function f for this expression can be defined over any domain.
Some examples, with their graphs are given.

1. $f(x) = x^2 + 3$ for $x \in \mathbb{R}$
(the symbol $\in$ means 'is a member of').
The range is $f(x) \geq 3$.



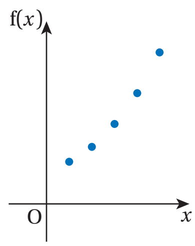
2. $f(x) = x^2 + 3$ for $x \geq 0$.
The range is also $f(x) \geq 3$.



Note

The point on the curve where $x = 0$ is included and this is denoted this by a solid dot.
If the domain were $x > 0$, then the point would not be part of the curve and this is indicated by a hollow dot.

3. $f(x) = x^2 + 3$ for $x = 1, 2, 3, 4, 5$.
The range is the set of numbers 4, 7, 12, 19, 28.



Note

This time the graphical representation consists of just five separate points.

Example 1

Question

The function, f, is defined by $f(x) = x^2$ for $x \leq 0$
and $f(x) = x$ for $x > 0$.

a Find f(4) and f(–4).

b Sketch the graph of f(x).

c Give the range of f.

Answer

a For $x > 0$, $f(x) = x$,
therefore $f(4) = 4$.
For $x \leq 0$, $f(x) = x^2$,
therefore $f(-4) = (-4)^2 = 16$.

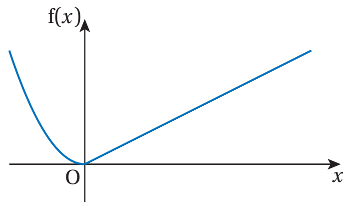
(continued)

(continued)

Answer

b To sketch the graph of a function, use what you know about lines and curves in the *xy*-plane.

So $f(x) = x$ for $x > 0$ is the part of the line $y = x$ which corresponds to positive values of x, and $f(x) = x^2$ for $x \le 0$ is the part of the parabola $y = x^2$ that corresponds to negative values of x.



c The range of f is $f(x) \ge 0$.

Exercise 1

1 Find the range of f in each of the following cases.

a $f(x) = 2x - 3$ for $x \ge 0$

b $f(x) = x^2 - 5$ for $x \le 0$

c $f(x) = 1 - x$ for $x \le 1$

d $f(x) = \dfrac{1}{x}$ for $x \ge 2$

2 Sketch the graph of each function given in question 1.

3 The function f is such that $f(x) = -x$ for $x < 0$

and $f(x) = x$ for $x \ge 0$.

a Find the value of f(5), f(–4), f(–2) and f(0).

b Sketch the graph of the function.

4 The function f is such that $f(x) = x$ for $0 \le x \le 5$

and $f(x) = 5$ for $x > 5$.

a Find the value of f(0), f(2), f(4), f(5) and f(7).

b Sketch the graph of the function.

c Give the range of the function.

1.2 Composite functions

Look at the two functions f and g given by $f(x) = x^2$ and $g(x) = \dfrac{1}{x}$ for $x \ne 0$.

When $g(x)$ replaces x in $f(x)$ this gives the **composite function**

$$f[g(x)] = f\left(\frac{1}{x}\right) = \frac{1}{x^2} \text{ for } x \ne 0$$

A composite function formed this way is also called a **function of a function** and it is denoted by fg.

For example, if $f(x) = 3^x$ and $g(x) = 1 - x$ then $gf(x)$ means the function g of $f(x)$.

$\Rightarrow \quad gf(x) = g(3^x) = 1 - 3^x$

Also $\quad fg(x) = f(1 - x) = 3^{(1-x)}$

This example shows that $gf(x)$ is *not* always the same as $fg(x)$.

Exercise 2

1. The functions f, g and h are defined by $f(x) = x^2$, $g(x) = \frac{1}{x}$ for $x \neq 0$ and $h(x) = 1 - x$. Find

   a $fg(x)$ **b** $fh(x)$ **c** $hg(x)$ **d** $hf(x)$ **e** $gf(x)$

2. When $f(x) = 2x - 1$ and $g(x) = x^3$ find the value of

   a $gf(3)$ **b** $fg(2)$ **c** $fg(0)$ **d** $gf(0)$

3. Given that $f(x) = 2x$, $g(x) = 1 + x$ and $h(x) = x^2$, find

   a $hg(x)$ **b** $gh(x)$ **c** $gf(x)$

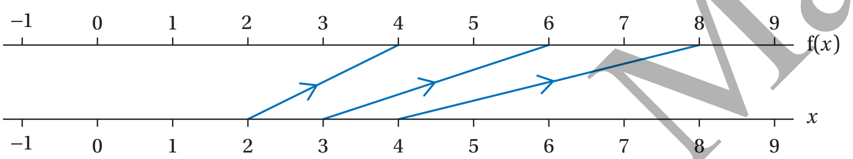
4. When $f(x) = \sin x$ and $g(x) = 3x - 4$ find

   a $fg(x)$ **b** $gf(x)$

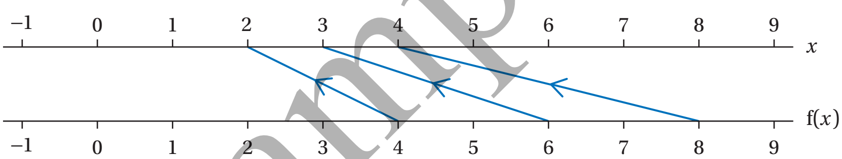
1.3 Inverse functions

Look at the function f where $f(x) = 2x$ for $x = 2, 3, 4$.

The domain of f is {2, 3, 4} and the range of f is {4, 6, 8}. The relationship between the domain and range is shown in the arrow diagram.



It is possible to reverse this process, so that each member of the range can be mapped back to the corresponding member of the domain by halving each member of the range.



This process can be expressed algebraically.

When $x = 4, 6, 8$, then $x \to \frac{1}{2}x$ maps 4 to 2, 6 to 3 and 8 to 4.

This reverse mapping is a function in its own right and it is called the **inverse function** of f where $f(x) = 2x$.

Denoting this inverse function by f^{-1} we can write $f^{-1}(x) = \frac{1}{2}x$ for $x = 4, 6, 8$.

The function $f(x) = 2x$ for $x \in \mathbb{R}$ also has an inverse function, given by $f^{-1}(x) = \frac{1}{2}x$ which also has domain $x \in \mathbb{R}$.

If a function g exists that maps the range of f back to its domain, then g is called the inverse of f and it is denoted by f^{-1}.

Summary

- A function f where $f(x)$ is any expression involving one variable which gives a single value of $f(x)$ for each value of x.
- The set of values which the variable in a function can take is called the domain of the function.
- For each domain, there is a corresponding set of values of $f(x)$. These are values which $f(x)$ can take for values of x in that particular domain. This set is called the range of the function.
- The composite function fg means that $g(x)$ replaces x in $f(x)$.
- If a function g exists that maps the range of f back to its domain, then g is called the inverse of f and it is denoted by f^{-1}.
- When curve $y = f(x)$ is reflected in the line $y = x$, the equation of the reflected curve is found by interchanging x and y in the equation $y = f(x)$.
- When the equation of the reflected curve is $y = g(x)$, g is called the inverse of f, so $g = f^{-1}$.
- The modulus of $f(x)$ is written as $|f(x)|$ and it equals the positive value of $f(x)$, whether $f(x)$ itself is positive or negative.
- A rational expression can be simplified by factorising the numerator and the denominator and then dividing both by any common factors.
- A proper fraction can be decomposed into partial fractions and the form of the partial fractions depends on the form of the factors in the denominator.
- A linear factor gives a partial fraction of the form $\frac{A}{ax+b}$.
- A repeated factor gives two partial fractions of the form $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$.
- When the fraction is improper it must first be expressed as the sum of a polynomial and a proper fraction, and can then be decomposed into partial fractions.

Review

1 The function f is defined by $f(x) = \sqrt{x-1}$ for $x > 1$.

a Find the range of f.

b Find the value of $f(10)$.

2 The function f is defined by

$$f(x) = \sin x \quad \text{for} \quad 0 \le x < \pi$$
$$f(x) = \pi - x \quad \text{for} \quad \pi \le x < 2\pi.$$

a Sketch the graph of $f(x)$ for $0 \le x < 2\pi$.

b Find the range of f.

3 The functions f and g are defined by $f(x) = \sin x$ and $g(x) = \sqrt{x}$ both for $x \ge 0$.

a Find $gf(x)$.

b State a domain of $gf(x)$ so that gf has real values.

4 **a** Solve the equation $|x+2| = 1 - x$.

b Show that there are no values of x for which $|x| + 1 = x - |x|$.

5 Describe a sequence of transformations that maps the graph of $y = 2^x$ to the graph of $y = 3 + 2^{-x}$.

6 Simplify

a $\dfrac{x^2 - 9}{2x - 6}$ b $\dfrac{4x^2 - 25}{4x^2 + 20x + 25}$

7 Express $\dfrac{x-3}{x+6}$ as a number plus a proper fraction.

8 Express $\dfrac{3x^2 - 5x + 1}{x+3}$ as a linear polynomial plus a proper fraction.

9 Express $\dfrac{x^3 - 4x^2 + 5}{x-1}$ as a quadratic polynomial plus a proper fraction.

10 Express in partial fractions.

a $\dfrac{4}{(2x+1)(x-3)}$ b $\dfrac{(3x-2)}{(x+1)(4x-3)}$ c $\dfrac{2t}{(t^2-1)}$

11 Express in partial fractions.

a $\dfrac{x+4}{(x+3)(x-5)}$ b $\dfrac{(2x-3)}{(x-2)(4x-3)}$ c $\dfrac{4x^2}{4x^2-9}$

12 Express in partial fractions.

a $\dfrac{3x}{2x^2 - 2x - 4}$ b $\dfrac{3x-1}{x^2(x-3)}$

Assessment

1 The function f is defined by $f(x) = \sqrt{x-1}$ for $x \geq 1$.

a State the domain and range of f and find $f^{-1}(x)$.

b Solve the equation $f^{-1}(x) = 2x$.

2 a Express $\dfrac{x^2}{x^2-4}$ as a linear function plus a proper fraction.

b Hence express $\dfrac{x^2}{x^2-4}$ in partial fractions.

3 a Describe a sequence of two transformations that maps the graph of $y = |x+1|$ to the graph of $y = 1 - |1+x|$.

b Sketch the graph of $y = 1 - |1+x|$.

c Find the coordinates of the points of intersection of the graphs of $y = |x+1|$ and $y = 1 - |1+x|$.

d Hence find the possible values of x for which $|x+1| > 1 - |1+x|$.

4 Express each rational function in partial fractions.

a $\dfrac{4}{x^2 - 7x - 8}$ b $\dfrac{2x-1}{(2x+1)(x-2)^2}$ c $\dfrac{3}{x(2x+1)}$

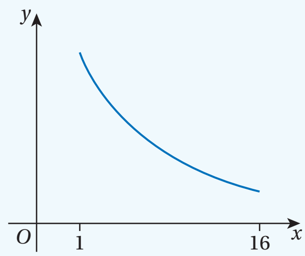
5 a Sketch the graph of $f(x) = \cos x$ for the domain $0 \leq x \leq 2\pi$.

b State the range of f.

c Given that $g(x) = 1 - |\cos x|$, find $fg(x)$.

d Find the value of $fg\left(\dfrac{\pi}{2}\right)$.

6 The curve with equation $y = \dfrac{63}{4x-1}$ is sketched below for $1 \le x \le 16$.



The function f is defined by $f(x) = \dfrac{63}{4x-1}$ for $1 \le x \le 16$.

a Find the range of f .

b The inverse of f is f^{-1} .

- **i** Find .
- **ii** Solve the equation $f^{-1}(x) = 1$.

c The function g is defined by $g(x) = x^2$ for $-4 \le x \le -1$

- **i** Write down an expression for $fg(x)$.
- **ii** Solve the equation $fg(x) = 1$.

AQA MPC3 January 2012

7 Express $\dfrac{16x}{(1-3x)(1+x)^2}$ in the form $\dfrac{A}{1-3x} + \dfrac{B}{1+x} + \dfrac{C}{(1+x)^2}$

AQA MPC4 June 2014 (part question)

8 **a** Sketch the curve with equation $y = 4 - |2x+1|$, indicating the coordinates where the curve crosses the axes.

b Solve the equation $x = 4 - |2x+1|$.

c Solve the inequality $x < 4 - |2x+1|$.

d Describe a sequence of two geometrical transformations that maps the graph of $y = |2x+1|$ onto the graph of $y = 4 - |2x+1|$.

AQA MPC3 June 2015