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1 Loci, Graphs and Algebra

Introduction

Polynomial functions always form a continuous curve with no breaks. However, when you divide one polynomial by another, the graph of the new function can have breaks in it and is said to be **discontinuous**. An example of such graphs is a **conic section**, which is the curve formed when a plane intersects a right circular cone. Some of the curves you meet the most in the real world are examples of conic sections, such as the ellipse that describes Earth's orbit around the Sun and the parabola that models the path of a football.

Objectives

By the end of this chapter, you should know how to:

- Sketch graphs of rational functions.
- Find equations of asymptotes to graphs.
- Solve inequalities involving rational functions.
- Describe and sketch various conic sections.
- Find the points of intersection between conic sections and coordinate axes and various straight lines.
- Find the Cartesian equation of simple loci that are described verbally.

Recap

You will need to remember how to . . .

- Solve construction problems involving loci.
- Solve equations, including quadratics.
- Sketch basic graphs such as $y = x^2$.
- Find the distance between two points in Cartesian coordinates.
- Solve simple inequalities such as $4x + 7 > 3(x - 4)$ and $x^2 - 7x + 10 \geq 0$.
- Transform graphs using stretches, reflections and translations.

1.1 Loci

In the context of graphs, a **locus** (plural **loci**) is a set of points that follow a given rule. Therefore, a locus can be represented by an equation.

For example, a locus is given as the points that are a distance of four units from the point (2, 3). This locus forms a circle with centre (2, 3). You know from previous studies that the equation of a circle is given in the form $(x - a)^2 + (y - b)^2 = r^2$, where r is the radius of the circle with centre (a, b). Therefore, the locus described above can be given as the Cartesian equation $(x - 2)^2 + (y - 3)^2 = 16$.

To find the Cartesian equation of a given locus, consider a general point on the curve (x, y) and use what you know about loci to help you formulate an appropriate equation.

Example 1

Question

Find the Cartesian equation of the locus of points that are equidistant from the point $(-1, 4)$ and the line $x = 2$.

Answer

(x, y) is a general point that obeys the rule.

Distance of (x, y) from line $x = 2$ is $|x - 2|$.

Distance from (x, y) to $(-1, 4)$ is

$\sqrt{(x+1)^2 + (y-4)^2}$

$(x+1)^2 + (y-4)^2 = (x-2)^2$

$x^2 + 2x + 1 + y^2 - 8y + 16 = x^2 - 4x + 4$

Equation of the locus is $(y-4)^2 = -6x + 3$.

Note

The two distances are equal.

Note

You will discover later in this chapter that this is a **conic equation** of a parabola.

Exercise 1

1. Find the Cartesian equation of the locus of points which are equidistant from the point $(3, -2)$ and the line $y = 5$.
2. Find the Cartesian equation of the locus of points which are a distance of four units from the point $(2, -3)$.
3. Find the Cartesian equation of the locus of points which are equidistant from the point $(-5, 3)$ and the line $x = 2$.
4. Find the Cartesian equation of the locus of points which are a distance of $4\sqrt{2}$ units from the point $(4, -4)$.

1.2 Rational functions

During your A-level Mathematics studies you will have learned how to sketch curves. In this chapter, you will learn how to sketch curves for functions that are more complicated than trigonometric or polynomial ones.

The graph of a rational function will always have a horizontal asymptote provided that the degree of x in the denominator is the same as or larger than the degree of x in the numerator. In this chapter, you will only deal with cases where there is a horizontal asymptote.

Sketching rational functions with a linear denominator

In order to sketch rational functions equations of the form $y = \frac{ax+b}{cx+d}$, you need to start by finding the asymptotes.

You should remember that in the context of sketching a curve, an **asymptote** is a line that becomes a tangent to the curve as x or y tends to infinity. (Vertical asymptotes are the lines where the graph is undefined.)

For example, take the curve of $y = \dfrac{4x-8}{x+3}$.

In order for $y \to \pm\infty$, the denominator of this function must tend to zero.

That is, as $x + 3 \to 0$, $x \to -3$. Hence, $x = -3$ is an asymptote.

To find the asymptote as $x \to \pm\infty$, you express the function as

$$y = \frac{4 - \dfrac{8}{x}}{1 + \dfrac{3}{x}}$$

As $x \to \pm\infty$, $\dfrac{3}{x} \to 0$ and $\dfrac{8}{x} \to 0$.

Therefore, $y \to \dfrac{4}{1} = 4$.

Tip

The asymptote occurs where the graph is undefined, so to find the vertical asymptote you need to equate the denominator of the rational function to zero.

Tip

Divide the top and bottom by x.

Hence, $y = 4$ is also an asymptote.

Notice that as $x \to \pm\infty$, the largest terms in the numerator and the denominator are $4x$ and x respectively, and so $y \approx 4x \div x = 4$.

$x = -3$ is a **vertical asymptote**, as it is parallel to the y-axis.

$y = 4$ is a **horizontal asymptote**, as it is parallel to the x-axis.

> **Tip**
>
> When the degrees of the numerator and denominator are the same, you can find the horizontal asymptote by dividing the leading coefficient of the numerator by the leading coefficient of the denominator.

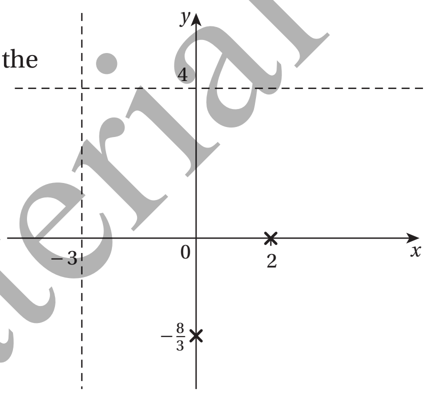
To be able to sketch $y = \frac{4x-8}{x+3}$, you also need to find where it crosses the x- and y-axes.

When $x = 0$: $y = -\dfrac{8}{3}$ When $y = 0$: $4x - 8 = 0 \quad \Rightarrow \quad x = 2$

In summary, you proceed as follows:

1. Draw the asymptotes using dashed lines.

2. Mark the points where the curve crosses the axes; as the numerator and the denominator of the function each contain only a linear term in x, the curve cannot cross either asymptote.

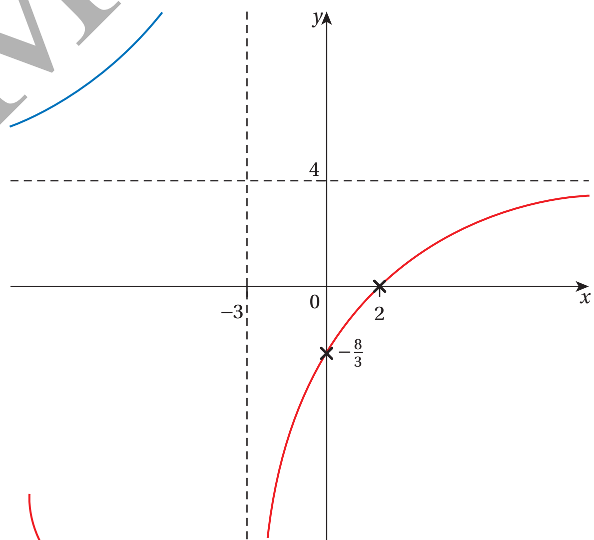


3. Considering the curve for $x > -3$, you can see that y tends to $-\infty$ as x approaches -3 from values of x greater than -3. Hence, the curve tends to $+\infty$ as x approaches -3 from values of x less than -3.

4. You can now complete the curve of $y = \dfrac{4x-8}{x+3}$.

To sketch the curve of a rational function in the form $y = \frac{ax+b}{cx+d}$:

1. **Find the vertical asymptote by equating the denominator to zero; find the horizontal asymptote by dividing the numerator and denominator by x, that is, expressing the function as $y = \frac{a+\frac{b}{x}}{c+\frac{d}{x}}$ and then considering what happens when x tends to infinity.**
2. **Substitute $x = 0$ and $y = 0$ into the function to find where the curve crosses the axes.**
3. **If necessary, consider the curve for the x-values, to see what happens to the y-values as x tends to $\pm\infty$.**

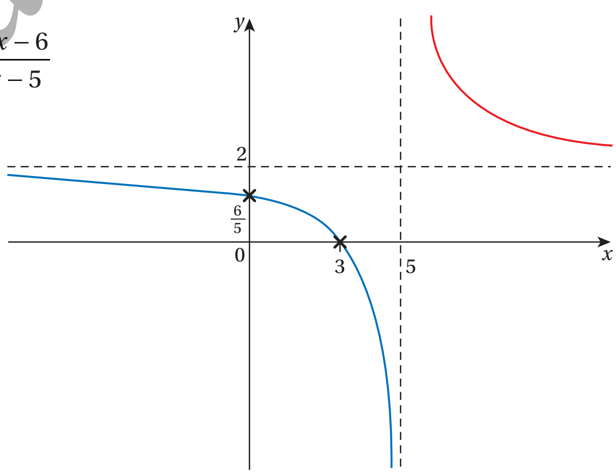


Example 2

Question

Sketch the graph of $y = \dfrac{2x-6}{x-5}$

> **Note**
>
> First, find the asymptotes.



Answer

Horizontal asymptote:
as $x \to \pm\infty$, $y \to \frac{2}{1}$, $y = 2$

Vertical asymptote:
as $y \to \pm\infty$, $x - 5 \to 0$, $x = 5$

When $x = 0$: $y = \dfrac{-6}{-5} = \dfrac{6}{5}$

> **Note**
>
> Next, find where the curve crosses the axes.

When $y = 0$: $2x - 6 = 0 \quad \Rightarrow \quad x = 3$

Note that the asymptotes here are parallel to the coordinate axes.

Note

Now copy and complete the sketch of $y = \frac{2x - 6}{x - 5}$ by also considering what will happen to the y-values as the x-values change.

Exercise 2

In questions **1**, **2** and **3**, state the equations of the asymptotes for the curve.

1. $y = \dfrac{x-1}{2x+6}$
2. $y = \dfrac{5}{x+2}$
3. $y = \dfrac{2x+7}{x-3}$
4. Show that $y = 2$ is an asymptote for the curve $y = \dfrac{2x+3}{x+4}$.
5. Show that $y = -4$ is an asymptote for the curve $y = \dfrac{8-4x}{x+3}$.
6. Sketch $y = \dfrac{6x-3}{x+4}$.
7. Sketch $y = \dfrac{3x-6}{x-1}$.
8. Sketch $y = \dfrac{2x+8}{3x-5}$, stating the equations of the asymptotes of the curve.

Sketching rational functions with a quadratic denominator

When sketching rational functions with equations of the form $y = \frac{ax^2+bx+c}{dx^2+ex+f}$, where both the numerator and the denominator are quadratics, you proceed as before by finding the asymptotes, where the graphs cross the axes, and consider the shape of the curve. However, with these functions, the shape of the curve requires consideration of stationary points and the number of asymptotes depends on the number of roots of the quadratic denominator.

- Two different roots will result in two vertical asymptotes.
- One repeated real root will result in one vertical asymptote.
- No roots indicates that there are no vertical asymptotes.

Regardless of the number of asymptotes, there may be values of y that are not realised by any value of x. Therefore, you need to find the **range** of the rational function in order to determine the set of possible values of y and in turn find the maximum and minimum points (the stationary points) of the curve.

Example 3

Question

Find the range of possible values of y when $y = \dfrac{3x-4}{x^2+3x-4}$.

Answer

Cross-multiplying,

$yx^2 + 3yx - 4y = 3x - 4$

$\Rightarrow \quad yx^2 + (3y-3)x + 4 - 4y = 0$

Therefore,

$(3y-3)^2 - 4y(4-4y) \geq 0$

Note

To find the range of values of y, you need to find the values for which $y = \frac{3x-4}{x^2+3x-4}$ has real solutions for x.

Note

For x to be real, we know that $b^2 - 4ac \geq 0$.

(continued)

In questions **5** to **7**, sketch the curve and clearly mark the points where the curve crosses the coordinate axes.

5 $\dfrac{x^2}{36}+\dfrac{y^2}{25}=1$

6 $\dfrac{(x-4)^2}{25}+\dfrac{(y-3)^2}{16}=1$

7 $\dfrac{(x-2)^2}{25}-\dfrac{(y+6)^2}{16}=1$

8 A straight line through $(1, 0)$ with gradient m intersects the hyperbola $\frac{x^2}{9}-\frac{y^2}{25}=1$ at point P. Show that the x-coordinate of point P satisfies the equation $(25-9m^2)x^2+18m^2x-(9m^2+225)=0$.

9 Write the asymptotes of $(x-5)(y+3)=6$, and sketch the curve.

10 An ellipse has the equation $\dfrac{x^2}{4}+\dfrac{y^2}{25}=1$.

a Sketch the ellipse.

b Given that the line $y=x+k$ intersects the ellipse at two distinct points, show that $-\sqrt{29}<k<\sqrt{29}$.

c The ellipse is translated by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ to form another ellipse whose equation is

$$25x^2+4y^2+50x-24y=c$$

Find the values of a, b and c.

Summary

- A locus is a set of points that obey a certain rule, and a locus can be expressed graphically, verbally or in the form of an equation.
- Asymptotes show the 'end behaviour' of a graph as x or $y \to \pm\infty$.
- To sketch the graphs of rational, parabola, ellipse and hyperbola equations you might need to find the:
  - Asymptotes (if applicable; parabolas and ellipses do not have asymptotes)
  - Intercepts with the axes, if any
  - Coordinates of any maxima or minima (if applicable).
- You can solve a rational inequality by:
  - Using algebra to multiply both sides of the inequality by $(cx+d)^2$
  - Sketching $y=\frac{ax+b}{cx+d}$, then solving $\frac{ax+b}{cx+d}=k$ and comparing the two results to find the solution.
- Conic sections are a family of curves with standard equations, and include the:
  - Parabola, $y^2=4ax$
  - Ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
  - Hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
  - Rectangular hyperbola, $xy=c^2$.

- The transformations of more complicated curves follow the same rules linear transformations do.

  If $y = f(x)$, then

  - $y = f(x) + a$ results in a positive translation in the y-direction
  - $y = f(x) - a$ results in a negative translation in the y-direction
  - $y = f(x + a)$ results in a negative translation in the x-direction
  - $y = f(x - a)$ results in a positive translation in the x-direction
  - $y = af(x)$ results in a stretch parallel to the y-axis, with scale factor a
  - $y = f(ax)$ is a stretch parallel to the x-axis, with scale factor $\frac{1}{a}$
  - $y = -f(x)$ is a reflection in the y-axis
  - $y = f(-x)$ is a reflection in the x-axis.

Review exercises

1. Find the Cartesian equation of the locus of points which are equidistant from the point $(5, -1)$ and the y-axis.
2. Sketch the graph of $y = \dfrac{2x^2 + 3x - 5}{x^2 - x - 2}$.
3. Sketch the graph of $y = \dfrac{3x^2 + 4x + 4}{x^2 - 2x - 3}$.
4. Solve $\dfrac{2x - 1}{x + 3} > 3$.
5. Solve $\dfrac{x^2 - x - 2}{x^2 + 3x + 2} > 1$.
6. Sketch $\dfrac{(x-3)^2}{36} - \dfrac{(y+2)^2}{25} = 1$.
7. The curve $x^2 + \frac{y^2}{9} = 1$ is translated by k units in the positive y-direction.

   a Show that the equation of the curve after this translation is $x^2 + \frac{(y-k)^2}{9} = 1$.

   b Show that if the line $x + y = 3$ intersects the translated curve, the y-coordinate of the points of intersection satisfies the equation $10y^2 - (54 + 2k)y + k^2 + 72 = 0$.

Practice examination questions

1. **a** **i** Write down the equations of the two asymptotes of the curve $y = \frac{1}{x-3}$. (2 marks)

   ii Sketch the curve $y = \frac{1}{x-3}$, showing the coordinates of any points of intersection with the coordinate axes. (2 marks)

   iii On the same axes, again showing the coordinates of any points of intersection with the coordinate axes, sketch the line $y = 2x - 5$. (1 mark)

   b **i** Solve the equation $\dfrac{1}{x-3} = 2x - 5$. (3 marks)

   ii Find the solution of the inequality $\dfrac{1}{x-3} < 2x - 5$. (2 marks)

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2 A parabola P has equation $y^2 = x - 2$.

a **i** Sketch the parabola P. (2 marks)

ii On your sketch, draw two tangents to P which pass through the point $(-2, 0)$. (2 marks)

b **i** Show that, if the line $y = m(x + 2)$ intersects P, then the x-coordinates of the points of intersection must satisfy $m^2x^2 + (4m^2 - 1)x + (4m^2 + x) = 0$. (3 marks)

ii Show that, if this equation has equal roots, then $16m^2 = 1$. (3 marks)

iii Hence find the coordinates of the points at which the tangents to P from the point $(-2, 0)$ touch the parabola P. (3 marks)

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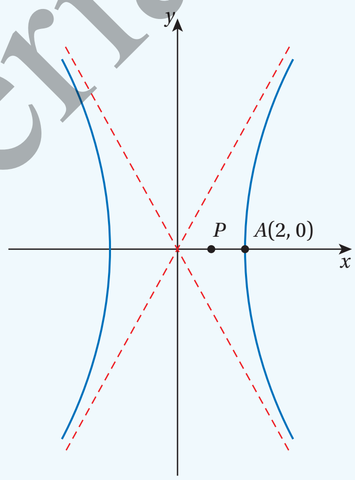
3 The diagram shows the hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and its asymptotes.

The constants a and b are positive integers.

The point A on the hyperbola has coordinates $(2,0)$.

The equations of the asymptotes are $y = 2x$ and $y = -2x$.



a Show that $a = 2$ and $b = 4$. (4 marks)

b The point P has coordinates $(1, 0)$. A straight line passes through P and has gradient m. Show that, if this line intersects the hyperbola, the x-coordinates of the points of intersection satisfy the equation

$(m^2 - 4)x^2 - 2m^2x + (m^2 + 16) = 0$ (4 marks)

c Show that this equation has equal roots if $3m^2 = 16$. (3 marks)

d There are two tangents to the hyperbola which pass through P. Find the coordinates of the points at which these tangents touch the hyperbola.

(No credit will be given for solutions based on differentiation.) (5 marks)

AQA MFP1 January 2010

4 A curve has equation

$$y = \frac{x^2 - 2x + 1}{x^2 - 2x - 3}$$

a Find the equations of the three asymptotes to the curve. (3 marks)

b **i** Show that if the line $y = k$ intersects the curve, then

$(k - 1)x^2 - 2(k - 1)x - (3k + 1) = 0$ (1 mark)

ii Given that the equation $(k - 1)x^2 - 2(k - 1)x - (3k + 1) = 0$ has real roots, show that $k^2 - k \geq 0$. (3 marks)

iii Hence show that the curve has only one stationary point and find its coordinates. (No credit will be given for solutions based on differentiation.) (4 marks)

c Sketch the curve and its asymptotes. (3 marks)

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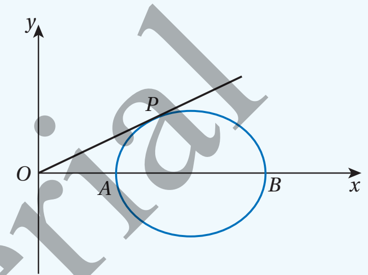
5 An ellipse is shown on the right.

The ellipse intersects the x-axis at the points A and B. The equation of the ellipse is $\frac{(x-4)^2}{4}+y^2=1$.



a Find the coordinates of A and B. (2 marks)

b The line $y = mx\ (m > 0)$ is a tangent to the ellipse, with point of contact P.

i Show that the x-coordinate of P satisfies the equation

$(1+4m^2)x^2-8x+12=0$ (3 marks)

ii Hence find the exact value of m. (4 marks)

iii Find the coordinates of P. (4 marks)

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6 The curve C has equation $y=\frac{x}{(x+1)(x-2)}$.

The line L has equation $y=-\frac{1}{2}$.

a Write down the equations of the asymptotes of C. (3 marks)

b The line L intercepts C at two points. Find the x-coordinates of these two points. (2 marks)

c Sketch C and L on the same axes.
(You are given that the curve C has no stationary points). (3 marks)

d Solve the inequality $\frac{x}{(x+1)(x-2)}\le-\frac{1}{2}$. (3 marks)

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